DIFFERENTIABILITY OF SAMPLE FUNCTIONS IN GAUSSIAN PROCESSES

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1. Let \{ξ(t), t∈T\}, T=[0, 1], be a Gaussian process, defined on an underlying probability space (X, Φ, P), in which X is the collection of all real valued functions x(t) defined on T, Φ is the σ-field of subsets of X generated by the Borel cylinders in X, and ξ(t) is defined by

\[ ξ(t, x) = x(t) \quad \text{for} \ t ∈ T, \ x ∈ X. \]

Here, by a Borel cylinder in X, we mean a subset BX of X defined by

\[ B_X = \{ x ∈ X; [x(t_1), \ldots, x(t_n)] ∈ B \} \]

where \( t_1, \ldots, t_n ∈ T \), and B is a Borel set in the n-dimensional Euclidean space. Such a process exists according to the Kolmogoroff Extension Theorem. If, further,

\[
E[ξ(t)] = 0 \quad \text{for} \ t ∈ T,
\]

and for some \( β, b > 0 \),

\[
E\{ |ξ(t') - ξ(t'')|^2 \} ≤ b |t' - t''|^β \quad \text{for} \ t', t'' ∈ T
\]

and the process is separable then²

\[ P^*(C_λ) = 1 \quad \text{for} \ 0 < λ < β/2 \]

where \( P^* \) is the outer measure of P and \( C_λ \) is the subset of X consisting of the Lipschitz \( λ \)-continuous elements, i.e., those functions which satisfy

\[ |x(t') - x(t'')| ≤ h |t' - t''|^λ \quad \text{for} \ t', t'' ∈ T \]

with \( h \) depending on \( x \).

In this article we consider the differentiability of the sample paths \( x(t) \) of the process \{ξ(t), t∈T\}. We shall assume that our process satisfies the additional condition that for some \( α, a > 0 \),

\[
E\{ |ξ(t') - ξ(t'')|^2 \} ≥ a |t' - t''|^α \quad \text{for} \ t', t'' ∈ T.
\]

Our result is stated in the theorem in §2, after a brief discussion of

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2 A proof for this statement can be found for instance in [4].
the measure induced on the space of continuous functions by the process. Our theorem implies in particular that, if $\alpha/2 < 1$, then almost every sample function $x(t)$ is almost nowhere differentiable with respect to $t$ on $T$. This is an extension of the classical result of Wiener for the Brownian motion process.

2. Let $C$ be the subset of $X$ consisting of the continuous functions. Now that $C_\lambda \subset C$ and $P^*(C) = 1$, the probability measure $P$ induces a measure $m_\sigma$ on $C$ according to a theorem by Doob (Theorem 1.1, [1]). Specifically this measure is obtained as follows. Let $\mathfrak{F}_C$ be the field of Borel cylinders in $C$. A Borel cylinder $B_C$ in $C$ is a subset of $C$ of the type

$$B_C = \{ x \in C; [x(t_1), \ldots, x(t_n)] \in B \} = B_X \cap C$$

where $t_1, \ldots, t_n \in T$, $B$ is a Borel set in the $n$-dimensional Euclidean space and $B_X$ is as defined previously. If we now define a set function $m_\sigma$ on $\mathfrak{F}_C$ by

$$(4) \quad m_\sigma(B_C) = P(B_X),$$

then $m_\sigma$ is well-defined according to the above quoted theorem by Doob and is in fact a measure on the field $\mathfrak{F}_C$. Finally by means of a Carathéodory extension we have a measure space $(C, \mathcal{C}^*, m_\sigma)$ in which $\mathcal{C}^*$ is the $\sigma$-field of the Carathéodory measurable sets. Our theorem can now be stated.

**Theorem.** Let $\{\xi(t), t \in T\}$, $T = [0, 1]$, be a separable Gaussian process satisfying the conditions (1), (2), and (3). Let $m_\sigma$ be the measure induced on the space $C$ of continuous functions $x(t)$ defined on $T$. Let $\lambda > \alpha/2$. Then for almost every $x \in C$ relative to $m_\sigma$

$$\limsup_{s \downarrow 0} \frac{|x(t + s) - x(t)|}{s^\lambda} = \infty,$$

$$\limsup_{s \downarrow 0} \frac{|x(t - s) - x(t)|}{s^\lambda} = \infty$$

for almost every $t \in T$ relative to the Lebesgue measure.

The proof of this theorem is based on a lemma which we prove in §3. The theorem itself is proved in §4.

3. **Lemma.** Let $\lambda > \alpha/2$. Then for every $t \in (0, 1)$ there exists a subset of $C$, $\Gamma_t \in \mathcal{C}^*$ with $m_\sigma(\Gamma_t) = 1$ such that every $x \in \Gamma_t$ satisfies (5) and (6) at $t$.

**Proof.** Let $h > 0$, $s > 0$ and
\[ \Gamma_{t, h, s} = \{ x \in C; \mid x(t + s) - x(t) \mid \leq hs^\lambda \}. \]

Then \( \Gamma_{t, h, s} \subseteq \mathbb{R}^C \) and, by (4) and the fact that \( \{ \xi(t), t \in T \} \) is a Gaussian process, we have

\[ m_G(\Gamma_{t, h, s}) = \frac{1}{(2\pi)^{1/2}\sigma_{t, t+s}} \int_{|\eta| \leq hs^\lambda} \exp \left\{ -\frac{\eta^2}{2\sigma_{t, t+s}^2} \right\} d\eta \]

where

\[ \sigma_{t, t+s}^2 = E\{ |\xi(t + s) - \xi(t)|^2 \} \geq a s^\alpha \]

according to (3). Thus

\[ m_G(\Gamma_{t, h, s}) \leq (2/\pi a)^{1/2} h s^{\lambda - \alpha/2}. \]

Since \( \lambda - \alpha/2 > 0 \), \( \lim_{s \to 0} m_G(\Gamma_{t, h, s}) = 0 \). Let \( \{ s_i \} \) be a sequence of positive numbers such that \( \sum_{j=1}^\infty s_j^{\lambda - \alpha/2} < \infty \) and consequently \( \sum_{j=1}^\infty m_G(\Gamma_{t, h, s_j}) < \infty \). By the Borel-Cantelli Theorem, \( m_G(\Gamma_{t, h}) = 1 \) where \( \Gamma_{t, h} = \lim_{j \to \infty} \inf \Gamma_{t, h, s_j} \). Thus, from the definition of \( \Gamma_{t, h, s} \),

\[ \lim_{s \downarrow 0} \sup \frac{|x(t + s) - x(t)|}{s^\lambda} \leq h \quad \text{for} \ x \in \Gamma_{t, h}. \]

Let \( \Gamma^+_t = \bigcap_{h=1}^\infty \Gamma_{t, h} \). Then \( m_G(\Gamma^+_t) = 1 \) and

\[ \lim_{s \downarrow 0} \sup \frac{|x(t + s) - x(t)|}{s^\lambda} = \infty \quad \text{for} \ x \in \Gamma^+_t. \]

Similarly, there exists a subset \( \Gamma^-_t \) of \( C \) with \( m_G(\Gamma^-_t) = 1 \) such that

\[ \lim_{s \downarrow 0} \sup \frac{|x(t - s) - x(t)|}{s^\lambda} = \infty. \]

We only have to take \( \Gamma_t = \Gamma^+_t \cap \Gamma^-_t \) to complete the proof of the lemma.

4. **Proof of the Theorem.** Consider the product measure \( m = m_G \times m_L \) on the product space \( C \times T \), where \( m_L \) is the Lebesgue measure on \( T \). Let \( \Gamma \) be the subset of \( C \times T \) consisting of those elements \((x, t)\) for which (5) and (6) hold. We show that \( \Gamma \) is a measurable subset of \( C \times T \) by showing that the two functions of \((x, t), \lim_{s \to 0} s^{-\lambda} |x(t + s) - x(t)| \) and \( \lim_{s \to 0} s^{-\lambda} |x(t - s) - x(t)| \), are measurable. We assume that each \( x \in C \) has been extended beyond \( T \) to be constant so that \( x(t) \) is continuous in an interval containing \( T \) in its interior. Now

\[ \lim_{s \downarrow 0} \sup \frac{|x(t + s) - x(t)|}{s^\lambda} = \lim_{k \to \infty} \left\{ \sup_{0 < s < 1/k} \frac{|x(t + s) - x(t)|}{s^\lambda} \right\}. \]
Let \( \{s_j\} \) be a countable dense subset of \((0, 1/k)\); so that from the continuity of \( s^{-\lambda}|x(t+s) - x(t)| \) as a function of \( s \) in \((0, 1/k)\), we have

\[
\sup_{0<s<1/k} \frac{|x(t + s) - x(t)|}{s^\lambda} = \sup_j \frac{|x(t + s_j) - x(t)|}{s_j^\lambda}.
\]

Since each \( s^{-\lambda}|x(t+s_j) - x(t)| \) is a measurable function of \((x, t)\) on \( C \times T \) so is \( \limsup_{s \downarrow 0} s^{-\lambda}|x(t+s) - x(t)| \). Similarly,

\[
\limsup_{s \downarrow 0} s^{-\lambda}|x(t - s) - x(t)|
\]

is measurable, and \( \Gamma \) is measurable.

Now for each \( t \in T \), let

\[
\Gamma(t) = \{x \in C; (x, t) \in \Gamma\}.
\]

For almost every \( t \), \( \Gamma(t) \) is a measurable subset of \( C \) and for every \( t \), \( \Gamma(t) \supseteq \Gamma_t \) of the Lemma. Since \( \mu_{\sigma}(\Gamma_t) = 1 = \mu_{\sigma}(C) \), we have

\[
\int_0^1 \mu_{\sigma}(\Gamma(t))\mu_L(dt) = \int_0^1 \mu_{\sigma}(\Gamma_t)\mu_L(dt) = 1
\]

and hence by Fubini’s theorem, \( \mu(\Gamma) = 1 \) and, furthermore,

\[
0 = \mu(C \times T - \Gamma) = \int_t \mu_L\{t \in T; (x, t) \in \Gamma\}\mu_{\sigma}(dx).
\]

Thus, for almost every \( x \in C \),

\[
\mu_L\{t \in T; (x, t) \in \Gamma\} = 0;
\]

that is, for almost every \( x \in C \),

\[
\mu_L\{t \in T; (x, t) \in \Gamma\} = 1.
\]

This completes the proof of the Theorem.

**Bibliography**


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