for \( n = k + 1 \) if it holds for \( n = k \). Hence, by induction, it holds for every \( n \).

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**References**


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**SOME GENERALIZATIONS OF OPIAL'S INEQUALITY**

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The inequality \( \int_0^a uu' \leq a/2\int_0^a |u'|^2 \) which is valid for absolutely continuous \( u \) with \( u(0) = 0 \) has received successively simpler proofs by Opial, [5], Olech [4], Beesack [1], Levinson [2], Pederson [6], and Mallows [3]. It is the purpose of this paper to use the method of Olech to obtain some more general inequalities.

**Theorem 1.** Let \( u \) be absolutely continuous on \((a, b)\) with \( u(a) = 0 \), where \( -\infty \leq a < b < \infty \). Let \( f(t) \) be a continuous, complex function defined for all \( t \) in the range of \( u \) and for all real \( t \) of the form \( t(s) = \int_0^s |u'(x)| \, dx \). Suppose that \( |f(t)| \leq f(|t|) \), for all \( t \), and that \( f(t_1) \leq f(t_2) \) for \( 0 \leq t_1 \leq t_2 \). Let \( r \) be positive, continuous and in \( L^{1-q}[a, b] \), where \( 1/p + 1/q = 1, p > 1 \). Let \( F(s) = \int_0^s f(x) \, dx, s > 0 \). Then

\[
\int_a^b |f(u)u'| \, dx \leq F\left[ \left( \int_a^b r^{1-q} \right)^{1/q} \left( \int_a^b |u'|^p \right)^{1/p} \right]
\]

with equality iff \( u(x) = A f_x r^{1-q} \). The same result (but with equality for \( u(x) = \int_x^b r^{1-q} \) holds if \( u(b) = 0 \) and \( -\infty < a < b \leq \infty \).

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Proof. Let \( z(x) = \int_a^x |u'| \),
\[
\int_a^b |f(u(x))u'(x)| \, dx = \int_a^b \left| f \left( \int_a^x u' \right) \right| u'(x) \, dx
\]
\[
\leq \int_a^b \left| f \left( \int_a^x u' \right) \right| u'(x) \, dx
\]
\[
= \int_a^b f(z)z' \, dx = F(z(b)),
\]
\[z(b) = \int_a^b z' = \int_a^b r^{-1/p} r^{1/p} \leq \left( \int_a^b r^{1-q} \right)^{1/q} \left( \int_a^b r \right)^{1/p}.
\]
The result follows with the observation that \( F \) is nondecreasing.

Example. Let \( f(t) = t^{p-1}, \, p > 1, \, u(a) = 0, -\infty < a < b < \infty \), then
\[
\int_a^b |u^{p-1}u'| \leq c p(1/p)
\]
with equality iff \( u(x) = A \int_a^x r^{1-q} \). For \( p = 2 \), we obtain essentially Bessel's generalization [1].

Example. Let \( f(t) = \sum_{n=0}^\infty a_n t^n \) be an absolutely convergent power series with radius of convergence \( R \). Let \( F(s) = \int_0^\infty \sum_{n=0}^\infty |a_n| x^n \, dx \). If \( \int_a^b |u'| < R \), then the theorem is true with this choice for \( f \) since \( \int_a^b f(u)u' \leq \int_0^\infty \sum_{n=0}^\infty |a_n| |u| r^n u' \) and the function \( g(t) = \sum |a_n| t^n \) has the properties \( |g(t)| \leq g(|t|) \), \( g(t_1) \leq g(t_2) \) when \( 0 \leq t_1 \leq t_2 \).

Corollary. If \( F(t+s) \geq F(t) + F(s) \) for \( t, s \geq 0 \) and if \( u(a) = u(b) = 0 \) for \( -\infty < a < b < \infty \), then \( \int_a^b |f(u)u'| \leq F \left[ \lambda \left( \int_a^b |u'|^p \right)^{1/p} \right] \) where \( \lambda = (\int_a^b r^{1-q})^{1/p} = (\int_a^b r^{1-q})^{1/p-1} \) and \( \xi \) is the uniquely determined point where the two integrals are equal. Equality holds iff

\[ u(x) = A \int_a^x r^{1-q}, \quad a \leq x \leq \xi \]
\[ = A \int_\xi^b r^{1-q}, \quad \xi \leq x \leq b. \]

Proof. We have
\[
\int_a^\xi f(u)u' \leq F \left[ \left( \int_a^\xi r^{1-q} \right)^{1/q} \left( \int_a^\xi r \right)^{1/p} \right]
\]
and
\[
\int_{\xi}^{b} \left| f(u) u' \right| \leq F \left[ \left( \int_{\xi}^{b} r^{1-q} \right)^{1/q} \left( \int_{\xi}^{b} r \left| u' \right|^{p} \right)^{1/p} \right].
\]
The result follows by adding, using the fact that \( F(s+t) \leq F(s) + F(t) \), and noting the choice of \( \xi \).

**Example.** For \( r=1, a \) and \( b \) finite, and \( f(t) = t^{p-1} \) for \( p > 1 \), \( \int_{a}^{b} \left| u^{p-1} u' \right| \leq (1/\rho) \left( (b-a)/2 \right)^{p-1} \int_{a}^{b} \left| u' \right|^{p} \) with equality for
\[
u(x) = A(x - a); \quad a \leq x \leq (a + b)/2,
= A(b - x); \quad (a + b)/2 \leq x \leq b.
\]

For \( p = 2 \) and \( a = 0 \) we obtain Opial's inequality.

**Example.** Let \( f(t) = t^{p-1} \) for \( p > 1 \) and \( u = v^{1/p} \), then
\[
\max_{t} \left| v(t) \right| \leq 1/2 \int_{a}^{b} \left| v' \right| \leq (2p)^{-p} (b - a)^{p-1} \int_{a}^{b} \left| v^{1-p} \right| \left| v' \right|^{p}
\]

**Theorem 2.** Let \( u, f, \) and \( r \) be as in Theorem 1. If \( 0 < 1, 1/\rho + 1/q = 1 \), \( u(a) = 0 \), \( -\infty \leq a < b < \infty \), then
\[
\int_{a}^{b} \left| u'/f(u) \right| \ dx \geq G \left[ \left( \int_{a}^{b} r^{1-q} \right)^{1/q} \left( \int_{a}^{b} r \left| u' \right|^{p} \right)^{1/p} \right],
\]
where \( G(s) = \int_{0}^{s} 1/f(x) \ dx \).

Equality holds iff \( u(x) = x^{r1-q} \). If \( u(b) = 0 \), \( -\infty < a < b \leq \infty \), the same result holds.

**Proof.**
\[
\int_{a}^{b} \left| u'/f(u) \right| = \int_{a}^{b} \left| u'/f(u) \right|/f \left( \int_{a}^{b} u' \right) \leq \int_{a}^{b} \left| u' \right|/f \left( \int_{a}^{b} u' \right) = \int_{a}^{b} z'/f(z) = G(z(b))
\]
\[
z(b) = \int_{a}^{b} z = \int_{a}^{b} r^{-1/p} r^{1/p} \geq \left( \int_{a}^{b} r^{1-q} \right)^{1/q} \left( \int_{a}^{b} r \left| u' \right|^{p} \right)^{1/p}.
\]
The result follows with the observation that \( G \) is nondecreasing.
Example. For $0 < p < 1$, $f(t) = t^{p-1}$;

$$\int_a^b |u^{p-1}u'| \geq (1/p) \left( \int_a^b r^{1-q} \right)^{p-1} \left( \int_a^b |u'|^p \right).$$

Taking $r = 1$, $p = 1/2$, $u = v^2$, we obtain

$$\int_a^b |vv'|^{1/2} \leq 2^{-1/2} (b - a)^{1/2} \int_a^b |v'|.$$

The next theorem is a generalization of a different sort which can easily be proven by the methods used above.

**Theorem 3.** If $u$ and $v$ are absolutely continuous for $-\infty \leq a < b < \infty$ and if $u(a) = v(a) = 0$, then

$$\int_a^b |uv'| + |vu'| \leq \left[ \int_a^b r^{-2} \int_a^b s^{-2} \int_a^b r^2 |u'|^2 \int_a^b s^2 |v'|^2 \right]^{1/2}$$

with equality iff $u(x) = Af(x)r^{-2}$ and $v(x) = Bf(x)s^{-2}$.

**References**