A CHARACTERIZATION OF DIFFERENTIAL OPERATORS

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In this paper we shall give an algebraic characterization of differential-difference operators, i.e. a characterization of those continuous linear operators from $\mathfrak{D}(R^n)$ into $L^2(R^n)$ which can be constructed by combining partial differential operators, multiplications by locally L^2 functions, and translation operators. The domain and range of the operator are relatively unimportant, since we are mainly interested in the form of the operator. But, for convenience, in this paper the symbols D or Δ will always denote continuous linear operators from $\mathfrak{D}(R^n)$ into $L^2(R^n)$.

For each b in R^n, define the operator $T(b, D)$ to be $e^{-ibx}De^{ibx}$, where e^{ibx} and e^{-ibx} denote the operators of multiplication by the exponential functions e^{ibx} and e^{-ibx} respectively. Define $T(D)$ to be the complex linear space of operators spanned by $\{T(b, D) \,|\, b \text{ in } R^n\}$. In this paper we shall prove the following theorem:

Let D be a continuous linear operator from $\mathfrak{D}(R^n)$ into $L^2(R^n)$. Then D is a differential-difference operator if and only if $T(D)$ is finite dimensional.

The proof of part of this theorem is immediate. (For notation, see Hörmander [1], particularly pages 9–10.) For suppose

$$D = \sum_{j=1}^{m} \psi_j \tau_{x_j} D^{\nu_j}$$

is a differential-difference operator with locally L^2 coefficients ψ_j. Then

$$T(b, D) = \sum_j \psi_j \exp(ibx_j)\tau_{x_j \nu_j}! \sum_{\beta \le \nu_j} \frac{(ib)^{\nu_j - \beta} D^\beta}{(\nu_j - \beta)!\beta!}$$

which is a linear combination of operators in the finite set

$$\{\psi_j \tau_{x_j} D^\beta \,|\, j = 1, 2, \cdots, m, \beta \le \nu_j\}.$$

Thus $T(D)$ is finite dimensional.

Now suppose that $T(D)$ is finite dimensional. In order to show that D is a differential-difference operator, we shall first prove some preliminary lemmas.

LEMMA 1. *Suppose $T(b, D) = D + \sum_{j=1}^{m} c_j(b)\psi_j$ where ψ_j denotes the operator of multiplication by the locally L^2 function ψ_j and the functions ψ_j are linearly independent. Then c_j $(j = 1, 2, \cdots, m)$ is a complex valued linear function on R^n.*

PROOF. We shall show c_j is additive and continuous at the origin. This implies that c_j is linear.

$$T(b_1 + b_2, D) = \exp(-ib_1x) \exp(-ib_2x) D \exp(ib_2x) \exp(ib_1x)$$
$$= \exp(-ib_1x)(D + \sum c_j(b_2)\psi_j)\exp(ib_1x)$$
$$= D + \sum_j (c_j(b_1) + c_j(b_2))\psi_j.$$

Also $T(b_1+b_2, D) = D + \sum c_j(b_1+b_2)\psi_j$. Thus c_j is additive.

Since the functions ψ_j are linearly independent, we can choose functions $g_1, \cdots, g_m$ in $\mathfrak{D}$ so that $\int \psi_j g_k = \delta_{jk}$. Thus for a suitably large sphere S, $c_j(b) = \int_S T(b, D)g_j - \int_S Dg_j$. Then $\lim_{b \to 0} e^{ibx}g_j$ converges in $\mathfrak{D}$ to g_j, so $\lim_{b \to 0} De^{ibx}g_j$ converges in L^2 to Dg_j. Since $e^{ibx} \to 1$ uniformly on S as $b \to 0$, $\int_S T(b, D)g_j$ converges to $\int_S Dg_j$ as $b \to 0$. Thus $\lim_{b \to 0} c_j(b) = 0$.

LEMMA 2. *Suppose that for each b in R^n and each f in $\mathfrak{D}$ we have $T(b, D)f = Df$. Then there is a locally L^2 function ψ so that $Df = \psi f$.*

PROOF. Let f and g be functions in $\mathfrak{D}$ such that g is identically one on the support of f. Define the function ψ on the set where $g = 1$ by $\psi = Dg$. We want to show $Df = \psi f$. Let P_k be a sequence of trigonometric polynomials such that gP_k converges to f in $\mathfrak{D}$. Since D commutes with multiplication by trigonometric polynomials, $gDP_kg = gP_kDg \to gf\psi$ in L^2. Since we can define ψ in this way on arbitrarily large spheres, we can define ψ on all of R^n.

LEMMA 3. *Suppose that the complex linear space spanned by $\{T(b, \Delta) - \Delta \mid b$ in $R^n\}$ is a finite dimensional space V of differential operators with locally L^2 coefficients. Then Δ is also a differential operator with locally L^2 coefficients.*

PROOF. First choose a basis of differential operators for the space V Look at the finite set of coefficient functions that appear in these operators and choose a maximal linearly independent subset $\{\psi_j\}$. Then any operator in V can be written uniquely as $\sum_{\alpha, j} c_\alpha^j \psi_j D^\alpha$ for some constants c_α^j. In general, there are functions c_α^j defined on R^n so that $T(b, \Delta) - \Delta = \sum c_\alpha^j(b)\psi_j D^\alpha$. We compute

$$T(b + b', \Delta) = T(b, T(b, \Delta))$$

$$= T(b, \Delta + \sum c_\alpha^j(b')\psi_j D^\alpha)$$

$$= \Delta + \sum_{\alpha,j} c_\alpha^j(b)\psi_j D^\alpha + \sum_{\alpha,j} c_\alpha^j(b')\psi_j \sum_{\gamma \leq \alpha} \alpha! \frac{(ib)^{\alpha-\gamma} D^\gamma}{(\alpha - \gamma)!\gamma!}$$

$$= \Delta + \sum_{\gamma,j} c_\gamma^j(b)\psi_j D^\gamma + \sum_{\alpha,j} c_\alpha^j(b')\psi_j \sum_{\gamma \leq \alpha} \alpha! \frac{(ib)^{\alpha-\gamma} D^\gamma}{(\alpha - \gamma)!\gamma!} \cdot$$

The coefficient of $\psi_j D^\nu$ is equal to

$$\sum_{\alpha \geq \gamma} c_\alpha^j(b') \frac{\alpha!}{(\alpha - \gamma)!\gamma!} (ib)^{\alpha-\gamma} + c_\gamma^j(b)$$

$$= \sum_{\alpha \geq \gamma} c_\alpha^j(b') \frac{\alpha!}{(\alpha - \gamma)!\gamma!} (ib)^{\alpha-\gamma} + c_\gamma^j(b') + c_\gamma^j(b) \cdot$$

This expression has to be symmetric in b and b', giving for each j and γ the set of equations

$$\sum_{\alpha > \gamma} c_\alpha^j(b') \frac{\alpha!}{(\alpha - \gamma)!\gamma!} (ib)^{\alpha-\gamma} = \sum_{\alpha > \gamma} c_\alpha^j(b) \frac{\alpha!}{(\alpha - \gamma)!\gamma!} (ib')^{\alpha-\gamma} \cdot$$

Now change notation on the right side of this equation by letting $\beta = \alpha - \gamma$. We get

$$(1) \quad \sum_{\alpha > \gamma} c_\alpha^j(b') \frac{\alpha!}{(\alpha - \gamma)!\gamma!} (ib)^{\alpha-\gamma} = \sum_{\beta > 0} c_{\beta+\gamma}^j(b) \frac{(\beta + \gamma)!}{\beta!\gamma!} (ib')^\beta.$$

The sum on the right is a finite sum, say over $\beta = \beta_1, \beta_2, \cdots, \beta_m$. Choose $b_1, \cdots, b_m$ so that the matrix $[b_{jk}] = [(ib_k)^{\beta_j}]$ is invertible. This is equivalent to saying that there exist numbers $B_{\beta,k}$ with the property

$$\sum_{k=1}^m (ib_k)^\alpha B_{\beta,k} = 1 \quad \text{if} \quad \alpha = \beta,$$
$$= 0 \quad \text{if} \quad \alpha \neq \beta.$$

By letting $b' = b_1, \cdots, b_m$ successively in equation (1) above we get for each γ and j a set of m equations which we can solve for $((\beta+\gamma)!/\beta!\gamma!)c_{\beta+\gamma}^j(b)$, finding it equal to

$$\sum_k B_{\beta,k} \sum_{\alpha > \gamma} c_\alpha^j(b_k) \frac{\alpha!}{(\alpha-\gamma)!\gamma!} (ib)^{\alpha-\gamma}.$$

Thus, if $\beta > 0$,

$$
\begin{aligned}
\overset{j}{c}_{\beta+\gamma}(b) &= \sum_{\alpha > \gamma} \frac{\beta!}{(\beta+\gamma)!} \sum_k B_{\beta,k} \overset{j}{c}_\alpha(b_k) \frac{\alpha!}{(\alpha-\gamma)!} (ib)^{\alpha-\gamma} \\
&= \sum_{\alpha > \gamma} \left[\sum_k B_{\beta,k} \overset{j}{c}_\alpha(b_k) \frac{\alpha!\beta!}{(\alpha+\beta)!} \right] \frac{(\alpha+\beta)!(ib)^{\alpha-\gamma}}{(\beta+\gamma)!(\alpha-\gamma)!} .
\end{aligned}
$$

(2)

Define $k_j(\alpha, \beta)$ to be the quantity within the brackets. Note that $k_j(\alpha, \beta)$ is defined only if $\alpha > 0$ and $\beta > 0$. If we take the special case $\gamma = 0$ we get, for $\beta > 0$

$$
\overset{j}{c}_\beta(b) = \sum_{\alpha > 0} [k_j(\alpha, \beta)] \frac{(\alpha+\beta)!}{\alpha!\beta!} (ib)^\alpha
$$

(3)

which says that $\overset{j}{c}_\beta$ is a polynomial and that $k_j(\alpha, \beta)$ is the coefficient of the monomial $((\alpha+\beta)!/\alpha!\beta!)(ib)^\alpha$ in the function $\overset{j}{c}_\beta$.

We want to show that $k_j(\alpha, \beta)$ depends only on j and $\alpha+\beta$. We shall show this first in the special case where there are multi-indices α_0, β_0, and γ such that $\beta = \beta_0 + \gamma$ and $\alpha = \alpha_0 - \gamma$. Using equation (2) twice

$$
\begin{aligned}
\overset{j}{c}_\beta(b) &= \sum_{\alpha > 0} [k_j(\alpha, \beta)] \frac{(\alpha+\beta)!}{\alpha!\beta!} (ib)^\alpha \\
&= \overset{j}{c}_{\beta_0+\gamma}(b) = \sum_{\alpha_0 > \gamma} [k_j(\alpha_0, \beta_0)] \frac{(\alpha_0+\beta_0)!}{(\beta_0+\gamma)!(\alpha_0-\gamma)!} (ib)^{\alpha_0-\gamma} \\
&= \sum_{\alpha_0 > \gamma} [k_j(\alpha_0, \beta_0)] \frac{(\alpha+\beta)!}{\alpha!\beta!} (ib)^\alpha .
\end{aligned}
$$

So $k_j(\alpha, \beta) = k_j(\alpha_0, \beta_0)$ since they both can be described as the coefficient of $((\alpha+\beta)!/\alpha!\beta!)(ib)^\alpha$ in $\overset{j}{c}_\beta$.

From this we can conclude that $k_j(\alpha_1, \beta_1) = k_j(\alpha_2, \beta_2)$ whenever we can find a third pair of nonzero multi-indices (α_3, β_3) so that $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \alpha_3 + \beta_3$, and we can compare β_3 to both β_1 and β_2 in the way we have just done. By simple arguments we can show that if the multi-index ν has $|\nu| \geq 3$ and $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \nu$ and if none of the multi-indices $\alpha_1, \alpha_2, \beta_1, \beta_2$ is equal to zero, then $k_j(\alpha_1, \beta_1) = k_j(\alpha_2, \beta_2)$.

Thus, for $|\nu| \geq 3$, we can define $k_j(\nu)$ as $k_j(\alpha, \beta)$ for any α and β with $\alpha + \beta = \nu$.

Let $\Delta' = \sum_j \psi_j \sum_{|\nu| \geq 3} k_j(\nu) D^\nu$. Then

$$
T(b, \Delta') - \Delta' = \sum_j \psi_j \sum_{|\nu| \geq 3} k_j(\nu) \sum_{\beta < \nu} \frac{\nu!}{(\nu-\beta)!\beta!} (ib)^{\nu-\beta} D^\beta
$$

By comparing this with equation (3) we can see that there are numbers $k_j'(\alpha, \beta)$ so that

$$T(b, \Delta - \Delta') - (\Delta - \Delta')$$

$$= \sum_j \psi_j \left[\sum_{|\alpha+\beta| \leq 2} k_j'(\alpha, \beta) \frac{(\alpha+\beta)!}{\alpha!\beta!} (ib)^\alpha D^\beta + c_0^j(b) \right], \qquad \alpha > 0.$$

The right side of this equation contains only partial differential operators of order one and multiplications by functions, so we can define functions d_k^j ($k = 0, 1, \cdots, n$), such that

$$T(b, \Delta - \Delta') - (\Delta - \Delta') = \sum_j \psi_j \left[d_0^j(b) + \sum_{k=1}^n d_k^j(b) \frac{\partial}{\partial x_k} \right].$$

Using the same trick as before:

$$T(b + b', \Delta - \Delta') = T\left(b, \Delta - \Delta' + \sum_j \psi_j \left(d_0^j(b') + \sum_k d_k^j(b') \frac{\partial}{\partial x_k} \right) \right)$$

$$= \Delta - \Delta' + \sum_j \psi_j \left[d_0^j(b) + d_0^j(b') \right.$$

$$+ \sum_k d_k^j(b') \frac{\partial}{\partial x_k} + \sum_k d_k^j(b) \frac{\partial}{\partial x_k} + \left. \sum_k (ib_k)(ib_k') \right]$$

where b_k is the kth coordinate of b. This has to be symmetric in b and b', so we get

$$\sum_k d_k^j(b) b_k' = \sum_k d_k^j(b') b_k.$$

Let y_k be the point in R^n that has kth coordinate 1 and all the other coordinates equal to zero. Pick b' to be $y_1, \cdots, y_n$ in succession. We get $d_k^j(b) = \sum_{p=1}^n d_p^j(y_k) b_p$ and thus, by letting $b = y_q$, $d_k^j(y_q) = d_q^j(y_k)$. Let $\Delta'' = (1/2) \sum_j \psi_j \sum \sum_{q,k} d_k^j(y_q)(\partial/\partial x_q)(\partial/\partial x_k)$. Then

$$T(b, \Delta'') = \Delta'' + \sum_j \psi_j \left[\tfrac{1}{2} \sum_k d_k^j(y_q) b_q \frac{\partial}{\partial x_k} \right.$$

$$+ \tfrac{1}{2} \sum_q d_k^j(y_q) b_k \frac{\partial}{\partial x_q} + \tfrac{1}{2} \sum_k \sum_q d_k^j(y_q) b_k b_q \Big]$$

$$= \Delta'' + \sum_j \psi_j \left[\sum_k d_k^j(b) \frac{\partial}{\partial x_k} + \tfrac{1}{2} \sum_k \sum_q d_k^j(v_q) b_k b_q \right].$$

Thus

$T(b, \Delta - \Delta' - \Delta'') - (\Delta - \Delta' - \Delta'')$

$$= \sum_j \psi_j \left[d_0^j(b) - \tfrac{1}{2} \sum_k \sum_q d_k^j(y_q) b_k b_q \right]$$

$$= \sum_j \psi_j [c_j(b)].$$

By Lemma 1, c_j is a linear function on R^n, i.e., for each j there are points P^j and Q^j in R^n such that $c_j(x) = \langle P^j, x \rangle + i \langle Q^j, x \rangle$. Define Δ''' to be $-i \sum_j \psi_j [(P_1^j + iQ_1^j)(\partial/\partial x_1) + \cdots + (P_n^j + iQ_n^j)(\partial/\partial x_n)]$. Then $T(b, \Delta''') = \Delta''' + \sum_j \psi_j [c_j(b)]$. This implies $T(b, \Delta - \Delta' - \Delta'' - \Delta''') - (\Delta - \Delta' - \Delta'' - \Delta''') = 0$. By Lemma 2, $\Delta - \Delta' - \Delta'' - \Delta'''$ is multiplication by a locally L^2 function. Since Δ', Δ'', and Δ''' are all differential operators with locally L^2 coefficients, so is Δ.

PROOF OF THEOREM. Since $T(D)$ is finite dimensional and since $\{ T(b, \cdot) \mid b \text{ in } R^n \}$ is a commuting family of mappings from $T(D)$ into itself, we can write $T(D)$ as a direct sum $V^1 \oplus \cdots \oplus V^k$ where each subspace V^j is invariant and indecomposable under this family of mappings. Also a basis for $T(D)$ can be found so that on V^j the mappings $T(b, \cdot)$ have matrices which are lower triangular with all diagonal elements equal, i.e., of the form

$$\begin{bmatrix} p_j(b) & & \\ & \ddots & 0 \\ * & & p_j(b) \end{bmatrix}$$

(see Jacobson [2, p. 134]). It is easily checked that $p_j(a + b) = p_j(a) p_j(b)$, and an argument similar to that in Lemma 1 will show that p_j is continuous. Thus p_j is an exponential function. We would like to show that p_j is bounded. Suppose the contrary: then there is a b in R^n so that $|p_j(b)| = \alpha > 1$. Let f be in $\mathfrak{D}$. Define, for m an integer, $f_m = \exp(imbx)f/\beta^m$, where $1 < \beta < \alpha$. Then f_m converges to zero in $\mathfrak{D}$. Let D_1^j be an element of V^j such that $T(D_1^j)$ is one dimensional. Then $T(mb, D_1^j)f = (p_j(b))^m D_1^j f$ and we have

$$\| D_1^j f_m \|_2 = \left\| \frac{\alpha^m}{\beta^m} e^{imbx} D_1^j f \right\|_2 = \left| \frac{\alpha}{\beta} \right|^m \| D_1^j f \|_2.$$

Since D_1^j is continuous, the left side of this equation must go to zero as m gets large. But the right side grows without bound, so we have a contradiction. Thus p_j is a bounded exponential; $p_j(x) = \exp(ib_j x)$ for some b_j in R^n. If we let W^j denote the space of operators of the form $\tau_{b_j} A$ where A is in V^j and τ_{b_j} denotes the operator of translation by

b_j, then W^j is invariant and indecomposable under the family of mappings $\{T(b, \cdot) | b \text{ in } R^n\}$ and there is a basis for W^j such that the matrix of $T(b, \cdot)$ on W^j is zero above the diagonal and identically one on the diagonal. Applying Lemma 3 to the basis elements of W^j in succession, first to the basis element D_1^j such that $T(D_1^j)$ is one dimensional, then to the element D_2^j such that $T(D_2^j)$ is two dimensional, etc., we see that W^j consists of differential operators. Thus V^j consists of differential-difference operators, which implies that every operator in $T(D)$, including D, is a differential-difference operator.

REFERENCES

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