A COMBINATORIAL PROBLEM IN THE 
k-ADIC NUMBER SYSTEM

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1. Introduction. Let $\mathbb{N}$ denote the set of all nonnegative integers. The elements in $\mathbb{N}$ are represented in the $k$-adic number system by strings of integers as $a_1a_2 \cdots a_p$, $0 \leq a_i \leq k - 1$. Define a multivalued function on $\mathbb{N}$ by

$$\Gamma(a_1a_2 \cdots a_p) = \{a_1 \cdots (a_v - 1) \cdots a_p; 1 \leq v \leq p, a_v \geq 1\}$$

and $\Gamma(0) = \emptyset$, the null set. Put $\alpha_k(a_1a_2 \cdots a_p) = \sum a_v$, $v = 1, 2, \cdots, p$ and $\alpha_k(S) = \sum \alpha_k(n_i)$, $n_i \in S$ if $S \subseteq \mathbb{N}$.

$S$ is said to be closed if $S \subseteq \mathbb{N}$ and $\Gamma S \subseteq S$. $S_n = \{0, 1, \cdots, n-1\}$ is closed. The problem is to determine the maximum of $\alpha_k(S)$ when $S$ ranges over all closed $S$ with $|S| = n$, i.e. with $n$ elements. Our main result (Theorem 1) is that the maximum is $\alpha_k(S_n)$.

If we put $B_k(n) = \alpha_k(S_n)$, we get as a corollary

$$B_k(m_1 + m_2 + \cdots + m_k) \geq \sum_{r=1}^{k} B_k(m_r) + \sum_{r=2}^{k} (v-1)m_r,$$

$$m_1 \geq m_2 \geq \cdots \geq m_k \geq 0.$$

It is interesting that Theorem 1 can be derived from this inequality. We have no independent proof of it, except for $k = 2$.

The asymptotic properties of the function $A_k(n) = B_k(n+1)$ were studied in [1] by R. Bellman and H. N. Shapiro. $A_2(n)$ appeared in connection with determinants in [2]. A result in that paper will be extended in our Theorem 2. We also note that there is some connection with the "detecting sets" studied in [3]. In fact, it was an attempt to extend the results in [3] which gave rise to the present problem.

2. Main results. In this section we shall derive the following theorem:

**Theorem 1.** If $S$ is closed and $|S| = n$, then $\alpha_k(S) \leq \alpha_k(S_n)$.

To simplify notations we shall omit the index "$k$" in the proofs.

Putting 0's in front of a string does not alter the integer represented by the string. Hence we can assume that all integers in $S$ are represented by strings of the same length $p = p(S)$.

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Given $S \subseteq N$, we shall define a set $S^c \subseteq N$, called the compression of $S$. Let $S_\nu$ denote the set of all integers $n \in S$ for which $\alpha(n) = \nu$. Let $S^c_\nu$ denote the set of the $|S_\nu|$ smallest nonnegative integers $n$ for which $\alpha(n) = \nu$. Then define $S^c$ as the union of the sets $S^c_\nu$, $\nu = 0, 1, 2, \cdots$. We note that

\begin{align}
(2.1) \quad |S^c| &= |S|, \\
(2.2) \quad \alpha(S^c) &= \alpha(S).
\end{align}

We shall prove a lemma:

**Lemma 1.** If $p(S) = 2$ and $S$ is closed, then $S^c$ is closed.

**Proof.** It is instructive to imagine the integers $a_1a_2 \in S$ as points with coordinates $(a_1, a_2)$ in a 2-dimensional coordinate-system.

If $a_1 \neq 0$ and $a_2 \neq 0$ for every $a_1a_2 \in S_\nu$ (or $S^c_\nu$), then

\begin{equation}
|T_{S_\nu}| \geq |S_\nu| + 1 \quad \text{and} \quad |T_{S^c_\nu}| = |S^c_\nu| + 1.
\end{equation}

This holds surely when $\nu \geq k$.

If there is one and only one integer $a_1a_2 \in S_\nu$ (or $S^c_\nu$) for which $a_1$ or $a_2 = 0$, then we find

\begin{equation}
|T_{S_\nu}| \geq |S_\nu| \quad \text{and} \quad |T_{S^c_\nu}| = |S^c_\nu|.
\end{equation}

From (2.3) and (2.4) we get in both cases

\begin{equation}
|T_{S^c_\nu}| \leq |T_{S_\nu}|.
\end{equation}

If there are two integers $a_1a_2$ for which $a_1$ or $a_2 = 0$ then $S_\nu = S^c_\nu$ and (2.5) holds with equality.

$S$ is closed if and only if $T_{S_\nu} \subseteq S_{\nu-1}$ for $\nu = 1, 2, \cdots$. Then we find by (2.1) and (2.5)

\begin{equation}
|T_{S^c_\nu}| \leq |S^c_{\nu-1}|, \quad \nu = 1, 2, \cdots.
\end{equation}

From this inequality it follows $T_{S^c_\nu} \subseteq S_{\nu-1}$ for $\nu = 1, 2, \cdots$. Hence $S^c$ is closed and the lemma is proved.

We shall prove a second lemma

**Lemma 2.** Assume $p = p(S) \geq 3$ for $S \subseteq N$, and that $b_1b_2 \cdots b_p \in S$, $a_i = b_i$ and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_p < b_1 \cdots b_{i-1}b_{i+1} \cdots b_p$ implies $a_1 \cdots a_p \in S$ for $i = 1, 2, \cdots, p$. Then $b_1b_2 \cdots b_p \in S$, $a_1a_2 \cdots a_p < b_1b_2 \cdots b_p$ and $a_1 + \cdots + a_p \leq b_1 + \cdots + b_p$ implies $a_1a_2 \cdots a_p \in S$.

**Proof.** We can assume $a_\nu \neq b_\nu$, $1 \leq \nu \leq p$. Then $a_1 < b_1$, since $a_1a_2 \cdots a_p < b_1b_2 \cdots b_p$. If there is $s \neq 1$ such that $a_s < b_s$, we get
From these inequalities we find \( a_1a_2 \cdots a_p \in S \) if \( b_1b_2 \cdots b_p \in S \).

Next we assume \( a_1 > b_1 \), for \( v > 1 \). Since \( a_1 + \cdots + a_p \leq b_1 + \cdots + b_p \), we get \( b_1 - a_1 \geq (a_2 - b_2) + \cdots + (a_p - b_p) \geq p - 1 \geq 2 \). Hence

\[
b_1b_2 \cdots b_p > (b_1 - 1)a_2b_3 \cdots b_p > (b_1 - 2)a_2 \cdots a_p \geq a_1a_2 \cdots a_p.
\]

Then from \( b_1b_2 \cdots b_p \in S \) we conclude \( a_1a_2 \cdots a_p \in S \).

**Proof of Theorem 1.** The proof is by induction over \( p = \rho(S) \).

If \( p = 1 \), \( S = S_1 \) and the theorem is true. Next we assume \( p = 2 \). The compressed set \( S^c \) is formed from \( S \). If \( S^c \neq S_1 \) let \( a_1a_2 \) be the smallest nonnegative integer not in \( S^c \) and let \( b_1b_2 \) be the largest integer in \( S^c \). Then we find \( a_1a_2 < b_1b_2 \), \( a_1 < b_1 \), \( a_2 > b_2 \), for \( S^c \) is closed by Lemma 1. We get even

\[
(2.6) \quad a_1 + a_2 > b_1 + b_2.
\]

For if \( a_1 + a_2 \leq b_1 + b_2 \), we can put \( c = a_1 + a_2 - b_2 \). Then \( a_1 < c \leq b_1 \) and \( cb_2 \in S^c \) for \( S^c \) is closed. Hence \( a_1a_2 \in S^c \), since \( a_1 + a_2 = c + b_2 \) and \( S^c \) is compressed. But \( a_1a_2 \in S^c \), and (2.6) follows by the contradiction.

If \( b_1b_2 \) is deleted from \( S^c \) and \( a_1a_2 \) is adjoined to it, we get a new closed and compressed set \( T \). We find by (2.1) and (2.2)

\[
\alpha(T) > \alpha(S).
\]

If \( T \neq S_1 \) we can find new integers \( a_1a_2 \) and \( b_1b_2 \). After a finite number of steps we get \( S_1 \), for the sum of all integers in the set is decreased at each step. By (2.7) the theorem holds for \( p = 2 \).

Now we assume that \( T \) is a closed set with \( p = \rho(T) \geq 3 \). For \( a_1 \) fixed we shall consider the set \( T(a_1) = \{ a_2a_3 \cdots a_p; a_1a_2 \cdots a_p \in T \} \). \( T(a_1) \) is closed and \( \rho(T(a_1)) = p - 1 \). By assumption the theorem holds for \( T(a_1) \). Replace \( T(a_1) \) by a set \( S_1; n = |T(a_1)| \), restore the digit \( a_1 \) and take union when \( a_1 = 0, 1, \cdots, k - 1 \). We get \( T_1 \) with \( \alpha(T_1) \geq \alpha(T) \). Note that \( |T(v - 1)| \geq |T(v)| \), since \( T \) is closed. It follows that \( T_1 \) is closed. Define \( T_1(a_2) = \{ a_2a_3 \cdots a_p; a_1a_2 \cdots a_p \in T_1 \} \). \( T_1(a_2) \) is closed. Replace it by a set of type \( S_1 \), restore the digit \( a_2 \) and take union when \( a_2 = 0, 1, \cdots, k - 1 \). \( T_2 \) is closed and \( \alpha(T_2) \geq \alpha(T_1) \). Continue with the digits \( a_3, \cdots, a_p, a_1, a_2, \cdots \). We get a sequence of closed sets: \( T, T_1, T_2, \cdots \), for which

\[
(2.8) \quad \alpha(T_{m+1}) \geq \alpha(T_m), \quad |T_m| = |T|.
\]

If \( T_{m+1} \neq T_m \), then the sum of all integers in \( T_{m+1} \) is smaller than the sum of all integers in \( T_m \). Hence there is an index \( q \) such that

\[
T_q = T_{q+1} = \cdots = T_{q+p}.
\]
Then we find that $T_q$ meets the requirements on $S$ in Lemma 2. If $T_q \neq S_n$, $n = |T|$, we can find a minimal $a_1 a_2 \cdots a_p \in T_q$ and a maximal $b_1 b_2 \cdots b_p \in T_q$ such that $a_1 \cdots a_p < b_1 \cdots b_p$ and, by Lemma 2, $a_1 + a_2 + \cdots + a_p > b_1 + b_2 + \cdots + b_p$.

We delete $b_1 b_2 \cdots b_p$ from $T_q$ and adjoin $a_1 a_2 \cdots a_p$ to the set. Then we get a closed set $U$ for which $\alpha(U) > \alpha(T_q)$. $U$ fulfills the requirements on $S$ in Lemma 2. If $U \neq S_n$ we proceed to a new closed set with larger $\alpha$-value. After a finite number of steps we get $S_n$. Hence $\alpha(T) \leq \alpha(S_n)$ and the theorem follows by induction over $p$.

It is interesting to know that Lemma 1 is not valid for $p(S) > 2$. This is seen by the example:

$$S = \{000, 001, 010, 100, 002, 011, 020, 110, 012, 021, 120\},$$
$$S^c = \{000, 001, 010, 100, 002, 011, 020, 101, 012, 021, 111\}.$$  

$S$ is closed, but $S^c$ is not closed since $110 \notin TS^c$ and $110 \in S^c$.

**Corollary.**

$$B_k(m_1 + \cdots + m_k) \geq \sum_{r=1}^k B_k(m_r) + \sum_{r=2}^k (\nu - 1)m_r,$$

$$m_1 \geq m_2 \geq \cdots \geq m_k \geq 0.$$

$$B_k(mn) \geq mB_k(n) + nB_k(m), \quad m, n \geq 1.$$  

**Proof.** Determine $p$ such that $m_1 \leq k^p$ and consider the set

$$S = \bigcup_{r=1}^k \{a_1 a_2 \cdots a_p(\nu - 1); a_1 a_2 \cdots a_p \in S_{m_r}\}.$$  

$S$ is closed and $|S| = m_1 + \cdots + m_k$. The first inequality follows if we determine $\alpha(S)$ and apply Theorem 1.

The second inequality follows if we determine $p$ and $q$ such that $m \leq k^p$ and $n \leq k^q$ and consider the set

$$T = \{a_1 \cdots a_p b_1 \cdots b_q; a_1 \cdots a_p \in S_m, b_1 \cdots b_q \in S_n\}.$$  

$T$ is closed, $|T| = mn$, $\alpha(T) = m\alpha(S_n) + n\alpha(S_m)$ and $\alpha(T) \leq \alpha(S_{mn})$.

3. Application to determinants. We assume here that $k = 2$. There is a one-one mapping from nonnegative integers to sets of nonnegative integers:

$$n = 2^n_1 + 2^n_2 + \cdots + 2^n_t \rightarrow N = \{n_1, n_2, \cdots, n_t\},$$

$$n_1 > n_2 > \cdots > n_t \geq 0,$$

$$0 \rightarrow \emptyset.$$
The set-theoretic counterpart to closed set of integers is closed family of sets: $\mathcal{F}$ is a closed family of sets if $N \subseteq \mathcal{F}, M \subseteq N$ implies $M \subseteq \mathcal{F}$.

Put $\alpha(N) = |N|$ and $\alpha(\mathcal{F}) = \sum \alpha(N), N \in \mathcal{F}$. For functions $f$ defined on a closed family $\mathcal{F}$, we put

\begin{equation}
\hat{f}(N) = \sum_{M \subseteq N} (-1)^{|M|}f(M),
\end{equation}

where the sum is taken over all subsets to $N$. It is easy to verify $(\hat{f})^\wedge = f$. The proof of the following lemma can also be omitted (cf. \[3, p. 481\]).

**Lemma 3.** If $f$ is defined on a closed family $\mathcal{F}$, and $M, N \in \mathcal{F}, M \subseteq N$,

\begin{equation}
\sum_{S \subseteq M} (-1)^{|S|}f(S \cap N) = 0.
\end{equation}

We shall prove the theorem on determinants:

**Theorem 2.** Let $N_1, N_2, \cdots, N_n$ be an enumeration of all sets in a closed family for which $N_i \cap N_j$ only if $i \leq j$. Then

\begin{equation}
|\hat{f}(N_i \cap N_j)|_{i,j=1}^n = \prod_{i=1}^{n} (-1)^{|N_i|}f(N_i).
\end{equation}

**Proof.** Multiply the last row in the determinant by $(-1)^{|N_n|}$. If $N_i \subseteq N_n$ we multiply the $i$th row by $(-1)^{|N_i|}$ and add to the last row. In the last row of the new determinant are all entries 0, except the last one which is $(\hat{f})^\wedge (N_n) = f(N_n)$. The value of the new determinant is $(-1)^{|N_n|}\left|\hat{f}(N_i \cap N_j)|_{i,j=1}^n = f(N_n)\right|\hat{f}(N_i \cap N_j)|_{i,j=1}^{n-1}$. If we note that $N_1 = \emptyset$ and $f(\emptyset) = f(\emptyset)$, the theorem follows by induction.

**Example.** Let $f(N) = 2^{|N|}$. Then $\hat{f}(M) = (-1)^{|M|}$. It follows that $2^{\alpha(\mathcal{F})}$ equals a determinant with all entries $+1$ or $-1$. If $\mathcal{F}$ is the family which corresponds to the integers $0, 1, \cdots, n$, we get Theorem 1 in [2].

**References**

