MEASURES THAT VANISH ON HALF SPACES

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I. Introduction. It is well known that if \( f \in L^1(E_n) \) (\( E_n \) denotes real Euclidean \( n \)-space, and all functions are complex valued) has the property that

\[
\int_H f(x) \, dx = 0
\]

for all half spaces \( H \), then \( f(x) = 0 \) a.e. It is natural to conjecture that if (1) holds for all \( H \in \mathcal{H}_n \), where \( \mathcal{H}_n \) is the set of all half spaces of \( E_n \) that exclude the unit sphere, then \( f(x) = 0 \) a.e. in \( \{ |x| = 1 \} \). Recently S. Helgason has proven this assuming the a priori estimate \( f(x) = O(|x|^{-m}) \) for all \( m > 0 \) [5]. The simple example (due to D. J. Newman) of \( f(x) = 1/(x_1 + ix_2)^3 \) if \( |x| \geq 1 \) and zero otherwise, which by Cauchy’s theorem satisfies (1) for all \( H \in \mathcal{H}_2 \), shows that without some assumption the conjecture is in fact false.

The purpose of this note is to characterize explicitly those \( f \in L^1(E_n) \) that satisfy (1) for all \( H \in \mathcal{H}_n \). The second section is devoted to a Paley-Wiener theorem for Hankel transforms which is needed in the proof of the main result. This is found in the final section together with a few concluding remarks. Another version of a Paley-Wiener theorem for Hankel transforms may be found in [4]. I am indebted to the referee for this reference.

II. A P-W Theorem for Hankel transforms. The following theorem of Plancherel and Polya [7] will be used.

\( (P) \) If \( f \in L^1(E_n) \) and \( F(y) = \int_{E_n} f(x) \exp(-2\pi i(x, y)) \, dx \) (or \( f \in L^2(E_n) \) and \( F \) its Fourier transform) then \( f \) vanishes a.e. in \( \{ |x| \geq 1 \} \) if and only if \( F \) is an entire function of exponential type \( 2\pi \) in every direction.

The Hankel transform of order \( \nu \) is defined by

\[
F(y) = \int_0^\infty f(x) J_\nu(2\pi xy) (xy)^{1/2} \, dx
\]

where \( J_\nu \) is the Bessel function of order \( \nu \) and either \( f \in L^1(0, \infty) \) or \( f \in L^2(0, \infty) \) and the integral is taken as l.i.m. [2, §42].

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Lemma 1. If $F$ is given by (2) and $\nu = n - \frac{1}{2}$ ($n = 1, 2, \cdots$) (resp. $\nu = n$) then $F$ (resp. $y^{1/2} F(y)$) is an entire function of exponential type $2\pi$ if and only if $f$ (resp. $x^{-1/2} f(x)$) is given by $f_0 + \sum_{j=1}^{\infty} c_j r_j$ where $f_0(x) = 0$ for $x \geq 1$, $r_j(x) = x^{-j/2}$ if $x \geq 1$ and zero otherwise, and the $c_j$'s are constants.

Proof. The proof for $\nu = n$ is just like the proof for $\nu = n - 1/2$ and so we confine our attention to this latter case. Furthermore, it is quite straightforward to reduce the case of $f \in L^1$ to $f \in L^2$ and thus it suffices to prove the lemma assuming that $f \in L^2$.

(a) Suppose first that $F$ has a zero of order $\geq n$ at $y > 0$. Then $G(y) = F(y)/y^n$ is also an entire function of exponential type $2\pi$ and setting $g(x) = f(x)/x^n$ we have

$$G(y) = y^{-n+1/2} \int_0^{\infty} g(x) J_n(2\pi xy) x^{n+1/2} d x.$$ 

If $g$ is considered as a radial function in $L^2(E_{2n+1})$ then except for a factor of $2\pi$ the right hand side of (3) gives the Fourier transform of $g$ [1, §2.6]. Applying (P) we conclude that in this case $f$ itself vanishes for $x \geq 1$.

(b) If $r_j$ is inserted for $f$ in (2) the corresponding $R_j$ are given by

$$R_j = y^{j-1} \int_0^{\infty} J_n(2\pi x) x^{-j+1/2} d x + S_j,$$

where $S_j$ is an entire function of exponential type $2\pi$ and vanishes at zero to order $\geq n$. From (a) we see that the first term cannot vanish and hence if a suitable linear combination of $r_j$ ($j = 1, 2, \cdots, n$) is subtracted from $f$ we are reduced to the situation in (a), and this completes the proof in one direction. The other direction is immediate since $J_n$ is entire of exponential type $2\pi$, a fact which we used in our assertion about $S_j$.

The following is an immediate consequence of the lemma and (P).

Corollary. If $f \in L^1(E_n)$ is a radial function ($f(x) = \tilde{f}(|x|)$) and

$$\tilde{F}(y) = 2\pi i^k y^{-n} \int_0^{\infty} \tilde{f}(w/y) J_{k+(n-1)/2}(2\pi w) w^{n/2} d w,$$

then $F$ is entire of exponential type $2\pi$ if and only if $f = f_0 + \sum_{j=1}^{\infty} c_j r_j$ ($n \geq 2, k > 0$).

III. The main theorem. Let

$$A_n = \{ f \in L^1(E_n) : (1) \text{ holds for all } H \in \mathfrak{S}_n \}.$$ 

Observe that (i) $A_n$ is a closed subspace of $L^1(E_n)$, and (ii) $A_n$ is rota-
tion invariant in the sense that \( R \in SO(n) \) and \( f \in A_n \) implies that \( Rf \in A_n(Rf(x) = f(Rx)) \). Denote by \( \sum f_k(|x|, x') \) \((x' = x/|x|)\) the expansion of \( f \) in spherical harmonics, i.e., for fixed \(|x|\), \( f_k \) is a spherical harmonic of degree \( k \) \([1, \S 2.7]\). The main reduction is accomplished by

**Lemma 2.** If \( f \in L^1(E_n) \) then \( f \in A_n \) if and only if \( f_k \in A_n \) for all \( k \).

**Proof.** Assume first that \( f \in A_n \), then \( f_k \) may be expressed as

\[
(4) \quad f_k(|x|, x') = \int_{SO(n)} Rf(x) Z_k(Ry', y') dR
\]

where \( y' \) is a fixed unit vector and \( Z_k \) is a zonal harmonic of degree \( k \) \([3, XI]\). Since translation is continuous in the \( L^1 \) norm, one verifies easily that \( R \to Rf \) is a continuous map from \( SO(n) \to L^1(E_n) \) and it then follows from (i) and (ii) that \( f \in A_n \).

Conversely, if \( f_k \in A_n \) for all \( k \) then so are the appropriate Abel means of \( \sum f_k \). Now the Abel means of a continuous function converge \([6]\)\(^2\) and thus by (i) \( f \in A_n \).

Next we identify the Fourier transform of \( A_n \) in

**Lemma 3.** If \( f \in L^1(E_n) \) and \( F(y) = \int_{E_n} f(x) \exp(-2\pi i(y, x)) dx \) then \( f \in A_n \) if and only if for all \( t \in E_n \) with \(|t| = 1\) we have that \( F_t \) is an entire function of exponential type \( 2\pi \) where \( F_t(z) = F(t_1z, t_2z, \cdots, t_nz) \).

**Proof.** By (ii) it suffices to consider \( t = (1, 0, \cdots, 0) \). Fubini's theorem yields

\[
(5) \quad F_t(z) = \int_{E_1} f(x) \exp(-2\pi izx_1) dx
\]

Since \( F \in A_n \), \( \{ \} \) as a function of \( x_1 \) vanishes for \(|x_1| \geq 1\) and thus \( F_t \) is entire of exponential type \( 2\pi \). The converse follows from (5) and (P).

**Theorem.** If \( f \in L^1(E_n) \) and \( f_k \) are given by (4) then \( f \in A_n \) if and only if

\[
f_k(|x|, x') = f_{k,0}(|x|, x') + \sum_{j=1}^{k-1} c_{k,j}(x')r_{k,j+1}(|x|)
\]

\(^*\) The theorem is given there for real valued continuous functions but may be easily extended to continuous functions with values in a Banach space (here \( L^1 \)).
where $f_{k,0}$ vanishes for $|x| \geq 1$. $c_{k,j}$ are harmonics of degree $k$ (as is $f_{k,0}$ for fixed $|x|$) and $r_m$ is defined in Lemma 1.

**Proof.** If $F_k$ denotes the Fourier transform of $f_k$ then

$$F_k(|y|, y') = 2\pi^k |y|^{-\frac{n}{2}} \int_0^\infty f_k(w/|y|, y') J_{k+(n-1)/2} (2\pi w) w^{n/2} dw$$

[2, §2.7]. The theorem now follows from Lemma 2-3 and the corollary of Lemma 1.

Helgason's result is obtained upon noticing that his a priori bound on $f$ carries over to $f_k$ and implies that the $c_{k,j}$ are identically zero. The results of this note carry over *mutatis mutandis* to measures, the details are omitted.

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**Bibliography**


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