Let $X$ be a compact metric space. It is known that if $U$ is the closed unit ball of $C_r(X)$ (the space of continuous real-valued functions on $X$ under the usual sup norm), a necessary and sufficient condition that $U$ be the closed convex hull of the set of its extreme points is that $X$ be totally disconnected (Bade [1]). It is also known (Phelps [4]) that if $C(X)$ is the space of all continuous complex-valued functions on $X$ under the sup norm, and if $U$ is the closed unit ball of $C(X)$, $U$ is always equal to the closed convex hull of the set of its extreme points (see also Goodner [2]). It is our purpose in this note to obtain information about $U$ in the case of $C(X)$ similar to that obtained for $C_r(X)$.

We make the following notational conventions: $D$ will denote the closed unit disc in the complex plane and $B$ will denote the set of points in $D$ of modulus 1. By $E$ we will mean the set of extreme points of $U$ (the closed unit ball of $C(X)$); $E$ is the set of all elements of $U$ which map $X$ into $B$. The topological dimension of $X$ as defined in Hurewicz and Wallman [3] will be denoted by $\dim X$.

Our theorem now reads as follows:

**Theorem.** Let $X$ be a compact metric space. Then the following are equivalent:

1. $\dim X \leq 1$;
2. $U$ is a subset of the convex hull of $E$.

**Proof.** We first observe that if $f$ is a continuous map of a topological space $Y$ into $D$ which omits the origin, then there are two continuous maps $f_1$ and $f_2$ of $Y$ into $B$ such that $f = (f_1 + f_2) / 2$. We now show that condition (1) implies condition (2). (I am indebted to the referee for strengthening and combining several arguments to give the following proof.)

Let $f$ be in $U$. By Theorem VI.1 of Hurewicz and Wallman, the origin is an unstable value of $f$; by Proposition B of the same section, there is a continuous function $h_1$ which omits the origin such that...

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(1) If \( |f(x)| \geq 1/3 \), then \( h_1(x) = f(x) \);
(2) if \( |f(x)| < 1/3 \), then \( |h_1(x)| < 1/3 \).

Put \( h_2 = 2f - h_1 \). Then it is clear that \( h_1 \) and \( h_2 \) are in \( U \).

Suppose \( |h_1(x)| > 3\varepsilon > 0 \) for all \( x \in X \). By the same results in [3], there is a continuous function \( g_2 \) such that \( g_2 \) omits the origin and such that

(3) If \( |h_2(x)| \geq \varepsilon \), then \( g_2(x) = h_2(x) \);
(4) if \( |h_2(x)| < \varepsilon \), then \( |g_2(x)| < \varepsilon \).

Put \( g_1 = 2f - g_2 \). Now it is easy to check that \( g_1 \) and \( g_2 \) are in \( U \); moreover \( g_1 \) omits the origin since \( |g_1(x) - h_1(x)| = |g_2(x) - h_2(x)| \leq 2\varepsilon \) for all \( x \in X \). By the remark at the beginning of the proof, \( g_1 \) and \( g_2 \) are in the convex hull of \( E \); hence \( f = (g_1 + g_2)/2 \) is in the convex hull of \( E \).

We now prove that condition (2) implies condition (1). By [3, Theorem VI, §4] it suffices to prove the following: Let \( C \) be a closed subset of \( X \). Then if \( f \) is a continuous map of \( C \) into \( B \), there is an extension of \( f \) to a continuous map of \( X \) into \( B \).

Hence, let \( C \) and \( f \) be as above. Using Tietze's theorem, we can extend \( f \) to a continuous \( \bar{f} \) from \( X \) into \( D \). If condition (2) holds, there is a probability measure \( \mu \) on \( U \) (even one with finite support) such that \( \mu(E) = 1 \) and such that \( L(\bar{f}) = \int L(g) d\mu(g) \) for all \( L \) in the (complex) dual of \( C(X) \). Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence dense in \( C \) and define linear functionals \( L_n \) on \( C(X) \) by \( L_n(h) = h(x_n) \) for \( h \) in \( C(X) \). Then for each \( n \) we have

\[
\bar{f}(x_n) = L_n(\bar{f}) = \int L_n(g) d\mu(g) = \int g(x_n) d\mu(g);
\]

we may divide to obtain

\[
1 = \int_{E} \frac{g(x_n)}{\bar{f}(x_n)} d\mu(g) \quad \text{for all } n.
\]

Since \( |\bar{f}(x_n)| = |g(x_n)| = 1 \) for all \( g \) in \( E \) and since \( \mu \) is a probability measure, it must be the case that

\[
\mu \left\{ g \in E : g(x_n) \neq \bar{f}(x_n) \right\} = 0 \quad \text{for each } n.
\]

Hence,

\[
\mu \left( \bigcup_{n=1}^{\infty} \left\{ g \in E : g(x_n) \neq \bar{f}(x_n) \right\} \right) = 0;
\]
it follows that there is a $g^*$ in $E$ such that $g^*(x_n) = f(x_n) = f(x_n)$ for all $n$; since $\{x_n\}$ is dense in $C$, $g^*(x) = f(x)$ for all $x$ in $C$. This $g^*$ is the desired extension of $f$ and the proof is thereby complete.

**Bibliography**


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