ON FILIPPOV'S IMPLICIT FUNCTIONS LEMMA

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In 1959 A. F. Filippov published a paper [1] containing a lemma designed for use in the study of optimal control problems. Stated somewhat imprecisely, let $k$ be a continuous function on a compact set $Q$ in a finite dimensional space and with values in a finite dimensional space, and let $(u'(t): a \leq t \leq b)$ be a function with values in $Q$ such that $k(u'(\cdot))$ is measurable; then there exists a measurable function $u$ from $[a, b]$ to $Q$ such that $k(u(t)) = k(u'(t))$.

For purposes of stochastic control theory it is desirable to extend this to allow arbitrary measure spaces, instead of intervals of real numbers (Kushner [4]); for purposes of the calculus of variations it is desirable to relax the requirement of compactness on $Q$. We do both of these ($Q$ may be any separable metric space), and at the same time we permit the values of $k$ to lie in any Hausdorff space; it costs nothing. In §2 we give an application to optimal control theory.

ADDED IN PROOF. In an abstract in Amer. Math. Monthly 73 (1966), p. 927, M. Q. Jacobs announces a generalization of Filippov's lemma. This generalization is a special case of our Theorem 1.

1. Definitions and first theorem. If $\mathcal{M}$ is a $\sigma$-ring of subsets of a set $M$, and $S$ is a topological space, a function $g: M_0 \rightarrow S$ from a set $M_0$ of the class $\mathcal{M}$ to $S$ is called measurable if the inverse image of every compact set in $S$ belongs to $\mathcal{M}$.

THEOREM 1. Let $M$ be a measure space, $A$ a Hausdorff space, and $Q$ a topological space which is the union of a countable number of compact metrizable subsets. Let $k: Q \rightarrow A$ be continuous, and $y: M \rightarrow A$ a measurable function such that $y(M) \subseteq k(Q)$. Then there exists a measurable function $u: M \rightarrow Q$ such that

$$k(u(x)) = y(x) \quad \text{for all } x \text{ in } M.$$  

Proof. We proceed through a number of special cases.

Case 1. Let $Q$ be a closed subset of the right half line $[0, \infty)$. This set $Q$ we also call $L$, for convenience later, and we construct a measurable function $T: M \rightarrow L$ such that $k \circ T = y$.

Received by the editors June 21, 1966.

1 This research was carried out while one author (E. J. McS.) was Principal Investigator on Army Research Office grant ARO-D-31-124-G662, and one (R. B. W. Jr.) was a National Science Foundation fellow.
For each nonnegative integer $q$ we partition $k(L)$ into a disjoint union of differences of compact sets, $B^q_j$, $j = 1, 2, \cdots$. We define the sets $B^q_j$ as follows:

$$B^q_j = k(L \cap [0, j \cdot 2^{-q}]) - k(L \cap [0, (j - 1) \cdot 2^{-q}]).$$

For each $x \in M$ and each nonnegative integer $q$, set

$$T_q(x) = \inf k^{-1}(B^q_j),$$

where $j$ is that integer for which $y(x) \in B^q_j$. This function is trivially measurable, since it takes on only the countable set of values $\inf k^{-1}(B^q_j)$, $j = 1, 2, \cdots$, and the inverse images of these sets $B^q_j$ are measurable.

We claim now that the $T_q$ are an increasing sequence converging to our desired function $T$.

The sequence is increasing since for each $x$ and each $q$ there are integers $i$, $j$ such that $y(x) \in B^q_j$ and $B^q_{j+1}$; in fact we have $i = 2j - 1$ or $i = 2j$. By definition

$$B^{q+1}_i \subseteq B^q_j.$$  

Since $T_{q+1}(x) = \inf k^{-1}(B^{q+1}_i)$ and $T_q(x) = \inf k^{-1}(B^q_j)$ we have $T_{q+1}(x) \geq T_q(x)$.

For each $x$, the sequence $T_q(x)$ is bounded above, for if $x \in k^{-1}(B^q_j)$ then $T_q(x) \leq j$ for all $q$. Hence $T_q$ converges to a measurable function $T$ with values in $L$ (since $L$ is closed).

We finally claim that $k \circ T = y$. If this is false, then for some $x \in M$ there exists an open subset $U$ of $A$ such that $k(T(x)) \in U$, $y(x) \notin U$. Since $k$ is continuous, $k^{-1}(U)$ contains some neighborhood of $T(x)$. Therefore there is some $q$ and some $j$ such that

$$T(x) \in L \cap [(j - 1) \cdot 2^{-q}, j \cdot 2^{-q}]$$

and

$$L \cap [(j - 1) \cdot 2^{-q}, j \cdot 2^{-q}] \subseteq k^{-1}(U).$$

This implies that $T_q(x) = \inf k^{-1}(B^q_j)$ for this $j$. (Otherwise, by (1), we would have for all $n \geq q$, $T_n(x) \leq (j - 1) \cdot 2^{-q}$, while $\lim_{n \to \infty} T_n(x) = T(x) > (j - 1)2^{-q}$.) Since for this $j$, $B^q_j \subseteq U$, we have, by the definition of $T_q$, that $y(x) \in B^q_j \subseteq U$, giving a contradiction. This completes the proof of Case 1.

Case 2. We now let $Q$ be any space such that there is a closed subset $L$ of $[0, \infty)$ and a continuous map $\phi : L \to Q$, taking $L$ onto $Q$. We then have the following picture:

$$L \xrightarrow{\phi} Q \xrightarrow{k} A \xleftarrow{y} M.$$
where $k$ and $\phi$ are continuous and $y$ is measurable, and $y(M) \subseteq k(Q) = k(\phi(L))$. By Case 1, there is a measurable function $T: M \to L$ so that $(k \circ \phi) \circ T = y$. We set $u = \phi \circ T$ and claim that this is our desired function. We have $y = k \circ u$ immediately, since $y = k \circ \phi \circ T$. $u$ is also measurable for, if $C$ is a compact subset of $Q$, $\phi^{-1}(C)$ is closed in $L$ and hence is a countable union of compact subsets of $L$ (namely the sets $L \cap [0, n] \cap \phi^{-1}(C)$). $T^{-1}(\phi^{-1}(C))$ is therefore measurable, and this is exactly the desired statement that $u^{-1}(C)$ is measurable.

Case 3. We now prove the theorem as stated. Let $K_1, K_2, \cdots$ be a sequence of compact metrizable sets whose union is $Q$. Since every compact metric space is the continuous image of the Cantor discontinuum ([2, Theorem 3.28]), for each positive integer $n$ there is a closed subset $L_n$ of $[2n-1, 2n]$ (a translate of the Cantor set) and a continuous function $\phi_n: L_n \to K_n$ whose range is $K_n$. Define $L = \bigcup L_n$, and define $\phi$ to be the function on $L$ which coincides with $\phi_n$ on $L_n$. Now the hypotheses of Case 2 are satisfied and the proof is complete.

2. An application. We give an application of Theorem 1 to optimal control theory. Let $B$ be a subset of $\mathbb{R}^n$ and $C^*$ a Hausdorff space, and let $f^1, \cdots, f^n$ be continuous real-valued functions on $R \times B \times C^*$. An admissible control function is a measurable function $v: [a, b] \to C^*$, where $[a, b]$ is an interval in $R$; a trajectory corresponding to this control is an absolutely continuous function $x: [a, b] \to B$ such that

$$x_i'(t) = f_i(t, x(t), v(t)) \quad (i = 1, \cdots, n)$$

for almost all $t$ in $[a, b]$. Two generalizations of this have been considered in optimal control theory and our application concerns the relation between them. For each $(t, x)$ in $R \times B$, let $K(t, x)$ be the smallest convex set in $\mathbb{R}^n$ that contains the set

$$\{(f^1(t, x, v), \cdots, f^n(t, x, v)) : v \in C^*\}.$$  

We almost, but not quite, follow J. Warga ([7], [4]) in defining a relaxed admissible curve to be an absolute continuous function $x: [a, b] \to B$ such that for almost all $t$ in $[a, b]$

$$x'(t) \in K(t, x).$$  

(Warga’s definition has the closure of $K(t, x)$ in the right member of (3), which in the especially important case of compact $C^*$ makes no difference.)

Let $\mathcal{D}$ be the set of probability measures defined on the $\sigma$-algebra of Borel subsets of $C^*$. A relaxed control function ([5], [6], [9]) is a func-
tion \((P_t: a \leq t \leq b)\) from \([a, b]\) to \(\varnothing\) such that for all bounded continuous functions \((\phi(t, v): a \leq t \leq b, v \in C^*)\) the function whose value at \(t\) is

\[
\int_{C^*} \phi(t, v) P_t(dv)
\]

is measurable on \([a, b]\). A trajectory corresponding to this relaxed control is an absolutely continuous function \(x: [a, b] \to B\) such that for almost all \(t\) in \([a, b]\) the functions \((f^i(t, x(t), v): v \in C^*)\) are \(P_t\)-integrable over \(C^*\), and

\[
x''(t) = \int_{C^*} f^i(t, x(t), v) P_t(dv).
\]

Since the point whose \(i\)th coordinate is the right member of (5) is in \(K(t, x(t))\), every trajectory corresponding to a relaxed control is a relaxed admissible curve. We now prove a partial converse.

**Theorem 2.** If \(C^*\) is the union of a countable set \(K_1, K_2, K_3, \ldots\) of compact metrizable sets, every relaxed admissible curve \((x(t): a \leq t \leq b)\) is a trajectory corresponding to a relaxed control function, and more specifically to a relaxed control function \((P_t: a \leq t \leq b)\) such that for each \(t\) there is a compact subset \(K_t\) of \(C^*\) for which \(P_t(C^* - K_t) = 0\).

This theorem generalizes Theorem 4.1 of [7].

For \(q = 1, 2, \ldots\), let \(\varnothing_q\) be the set of those probability measures \(P\) on Borel subsets of \(C^*\) for which \(P(C^* - K_q) = 0\). Then the union \(\varnothing_0\) of the \(\varnothing_q\) is contained in \(\varnothing\).

For each \(q\) there is a countable set \(\{\phi_{q,1}, \phi_{q,2}, \ldots\}\) of continuous functions from \(K_q\) to \(R\) which is dense in the unit ball of the space \(C(K_q)\) of all such functions [3, p. 245]. For each pair \(P', P''\) of members of \(\varnothing_q\) we define

\[
\rho_q(P', P'') = \sum_{j=1}^{\infty} 2^{-j} \left| \int_{K_q} \phi_{q,j}(v) P'(dv) - \int_{K_q} \phi_{q,j}(v) P''(dv) \right|.
\]

This is a metric on \(\varnothing_q\). Convergence of \(\rho_q(P^{(n)}, P^{(0)})\) to 0, \((P^{(n)}, P^{(0)}) \in \varnothing_q\) is equivalent to convergence of \(\int_{K_q} \phi(v) P^{(n)}(dv)\) to \(\int_{K_q} \phi(v) P^{(0)}(dv)\) for all \(\phi\) in the set \(\{\phi_{q,1}, \phi_{q,2}, \ldots\}\), hence for all \(\phi\) in \(C(K_q)\). Given any sequence \(P^{(1)}, P^{(2)}, \ldots\) in \(\varnothing_q\), by the diagonal process we can choose a subsequence (for which we retain the same notation) such that the limits

\[
\lim_{n \to \infty} \int_{K_q} \phi_{q,j}(v) P^{(n)}(dv) \quad (j = 1, 2, 3, \ldots)
\]
exist. It follows at once that the limit

$$I(\phi) = \lim_{n \to \infty} \int_{K_q} \phi(v) P_n(dv)$$

exists for all $\phi$ in $C(K_q)$; it is linear, is nonnegative, if $\phi \geq 0$, and is 1 if $\phi = 1$. Hence by the Riesz representation theorem there is a measure $P^{(0)}$ in $\mathcal{Q}_q$ for which

$$I(\phi) = \int_{K_q} \phi(v) P^{(0)}(dv).$$

Then $\lim \rho_q(P^{(n)}, P^{(0)}) = 0$, so $\mathcal{Q}_q$ is compact.

We shall need the following fact.

(6) If $\phi: [a, b] \times K_q \to \mathbb{R}$ is continuous, the function $\Phi_\phi: [a, b] \times P_q \to \mathbb{R}$ whose value at $(t, P)$ is $\int_{K_q} \phi(t, v) P(dv)$ is continuous.

Since $\phi$ is uniformly continuous, $\Phi_\phi(t, P')$ is uniformly continuous on $[a, b]$ for each fixed $P'$ in $\mathcal{Q}_q$. By definition of $\rho_q$, it is continuous in $P'$ for each fixed $t$ in $[a, b]$. Hence (6) holds.

Now we topologize $\mathcal{Q}_0$ with the topology generated by the $\mathcal{Q}_q$. A set $G \subseteq \mathcal{Q}_0$ is open if and only if $G \cap \mathcal{Q}_q$ is an open subset of $\mathcal{Q}_q$ for $q = 1, 2, 3, \cdots$. Then a function on $\mathcal{Q}_0$ is continuous if and only if its restriction to each $\mathcal{Q}_q$ is continuous on $\mathcal{Q}_q$. In particular, from (6) we obtain

(7) If $\phi: [a, b] \times C^* \to \mathbb{R}$ is continuous, the function $\Phi_\phi: [a, b] \times \mathcal{Q}_0 \to \mathbb{R}$ whose value at $(t, P)$ $(t \in [a, b], P \in \mathcal{Q}_0)$ is $\int_{C^*} \phi(t, v) P(dv)$ is continuous.

Let $x: [a, b] \to \mathbb{R}$ be a relaxed admissible curve. There is a set $M$, consisting of almost all points of $[a, b]$, such that (3) holds for all $t$ in $M$. We apply Theorem 1, letting $A$ be $\mathbb{R}^{n+1}$ and $Q$ be $[a, b] \times \mathcal{Q}_0$, and defining $k$ to be the function whose value at $(t, P)$ is

$$k(t, P) = \left( t, \int_{C^*} f_1(t, x(t), v) P(dv), \cdots, \int_{C^*} f_n(t, x(t), v) P(dv) \right).$$

By (7), this is continuous on $Q$. For each $t$ in $M$, the point $x'(t)$ of $K(t, x)$ is the weighted mean of finitely many points of the set (2), so that (5) holds with a $P'$ concentrated on a finite subset of $C^*$, which is in $\mathcal{Q}_q$ for all large $q$. So if we define

$$y(t) = (t, x'(t), \cdots, x^{(n)}(t)) \quad (t \in M)$$

we have $y(M) \subseteq k(Q)$. Clearly $y$ is measurable, so by Theorem 1 there is a measurable function $u: M \to Q$ (whose value at $t$ we denote by $(\tau(t), P_i)$) such that $k(u(t)) = y(t)$ on $M$; that is,

$$\tau(t) = t, \int_{C^*} f_i(t, x(t), v) P_i(dv) = x^i(t) \quad (t \in M, i = 1, \cdots, n).$$
We complete this by letting $P_t$ be any measure in $\mathcal{P}_0$ on $[a, b] - M$.

If $A$ is a closed subset of $R$, by (7) $\Phi^{-1}_t(A)$ is a closed subset of $Q$, hence is a countable union of compact sets. Therefore $u^{-1}(\Phi^{-1}_t(A))$ is a measurable set, and $\Phi_t \circ u$ is measurable. That is, (4) is measurable, so $(P_t: a \leq t \leq b)$ is a relaxed control function and $x$ is a trajectory corresponding to it.

3. A generalization. If we permit the continuum hypothesis to be invoked, we can generalize Theorem 1 as follows.

**Theorem 4.** Let $M$ be a measure space, $A$ a Hausdorff space, and $Q$ a separable metric space. Let $k: Q \to A$ be continuous, and $y: M \to A$ a measurable function such that $y(M) \subseteq k(Q)$. Then (assuming the continuum hypothesis) there is a measurable function $u: M \to Q$ such that $y = k \circ u$.

Let $v_1, v_2, \ldots$ be a countable dense subset of $Q$. We map the set of closed subsets of $Q$ into the set of all sequences of real numbers thus: to a closed set $F$ in $Q$ there corresponds the sequence $(d_1, d_2, \ldots )$ where $d_n$ is the distance of $v_n$ from $F$. This map is one-to-one and the cardinal of the set of sequences is the cardinal $c$ of the continuum, so there are at most $c$ closed subsets of $Q$.

By the continuum hypothesis, there is an ordinal $\Omega$ which is preceded by $c$ ordinals, while if $\alpha < \Omega$, $\alpha$ is preceded by countably many ordinals. We can therefore label all compact subsets of $Q$ with a subset of the ordinals less than $\Omega$. Without loss of generality we may suppose that there is an ordinal $\Omega' \leq \Omega$ and for every ordinal $\alpha < \Omega'$ a compact set $C_\alpha$ such that the $C_\alpha$ ($\alpha < \Omega'$) are all the compact subsets of $Q$.

Now for each $\alpha < \Omega'$ we define $Q_\alpha = \bigcup_{\beta \leq \alpha} C_\beta$. This is the union of countably many compact sets. For each $\alpha < \Omega'$ define $M_\alpha$ to be the set of all $x \in M$ such that $y(x) \in k(Q_\alpha)$ and $y(x) \in k(Q_\beta)$ for all $\beta < \alpha$. These sets are disjoint, and their union is $M$. Since $Q_\beta$ and $\bigcup_{\beta < \alpha} C_\beta$ are countable unions of compact sets, $M_\alpha$ is measurable. By Theorem 1 there is a measurable function $u_\alpha: M_\alpha \to Q_\alpha$ such that $k(u_\alpha(x)) = y(x)$ for $x \in M_\alpha$. We define $u: M \to Q$ as follows: $u(x) = u_\alpha(x)$ if $x \in M_\alpha$. Since the $M_\alpha$ are disjoint, this is unambiguous.

We need only check that $u$ is measurable. If $K$ is a compact subset of $Q$ then $K = C_\alpha$ for some $\alpha$ (fixed hereafter). Then $K \subseteq Q_\alpha$ and

$$u^{-1}(K) = \bigcup_{\beta \leq \alpha} u^{-1}_\beta(K \cap Q_\beta).$$

Since the $u_\beta$ are measurable functions and this is a countable union,
it suffices to show that $K \cap Q_\beta$ is a countable union of compact sets for all $\beta \leq \alpha$. This, however, is true since

$$K \cap Q_\beta = \bigcup_{j \leq \beta} K \cap C_j.$$ 

This completes the proof of Theorem 4.

The proof given above actually applies to a larger class of spaces $Q$. $Q$ could be any topological space with a family of compact metrizable subsets with cardinality at most $c$ such that the union of the subsets in this family is all of $Q$ and such that any compact subset of $Q$ lies in the union of a countable subfamily.

**Bibliography**


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