A TWO-PARAMETER HOMOGENEOUS MEAN VALUE\(^1,2\)

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1. Introduction. The mean of order \( t \) of the positive values \( x = (x_1, x_2, \ldots, x_n) \) with positive weights \( w = (w_1, w_2, \ldots, w_n) \), \( \sum w_i = 1 \), is defined [3], [5] by

\[
M_t(x, w) = \left[ \sum_{i=1}^{n} w_i x_i^t \right]^{1/t}, \quad t \neq 0,
\]

(1.1)

\[
M_0(x, w) = \prod_{i=1}^{n} x_i^{w_i} = \lim_{t \to 0} M_t(x, w).
\]

Homogeneity in \( x \) distinguishes \( M_t \) from all other means of the form \( \phi^{-1} \left[ \sum_{i=1}^{n} w_i \phi(x_i) \right] \), where \( \phi \) is any function with a unique inverse \( \phi^{-1} \) [5, Theorem 84].

Without losing homogeneity, \( M_t \) has been generalized to the hypergeometric mean value \( M(t, c; x; w) \) [4]. In the present paper we shall make a further generalization while maintaining homogeneity. We construct a two-parameter mean, \( L(s, t; x) \), by first forming the mean \( M_s(x, w) \). Using an arbitrary weight function, \( P(u) \), we then take an integral average over all possible choices of the weights \( u \) satisfying \( \sum_{i=1}^{n} u_i = 1 \). Because \( u_n = 1 - u_1 - \cdots - u_{n-1} \), the average requires an \((n-1)\)-fold integration with respect to \( u_1, u_2, \ldots, u_{n-1} \).

If the variables \( x \) are all positive, then for any real \( s \) and \( t \) we define

\[
L(s, t; x) = \left[ \int_E M_s(x, u) P(u) du' \right]^{1/t}, \quad t \neq 0,
\]

(1.2)

\[
L(s, 0; x) = \lim_{t \to 0} L(s, t; x),
\]

where \( u' = (u_1, \ldots, u_{n-1}) \), \( du' = du_1 du_2 \cdots du_{n-1}, \) \( P(u) \geq 0, \int_E P(u) du' = 1 \) and \( E = \{u' | u_i > 0, 1 \leq i \leq n-1 \text{ and } u_n = 1 - u_1 - \cdots - u_{n-1} > 0\} \).

The \( L(s, t; x) \) mean is homogeneous in \( x \). It can be regarded as a special case of the integral mean [5, Chapter 3].

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but it is a generalization of previously known ways of constructing a homogeneous mean of the discrete variables \( x_1, x_2, \ldots, x_n \). After defining a set of natural weights \( w \) associated with the function \( P(u) \), we shall show (Theorem 3) that \( L(s, t; x) \) contains \( M_t(x, w) \) in the special case \( s = t \). It contains also the hypergeometric mean, \( M(t, c; x; w) \), in the special case \( s = 1 \) and \( P(u) = P(cw; u) \), a particular weight function depending on the parameters \( cw = (cw_1, cw_2, \ldots, cw_n) \) [4, Equation (2.2)]. All properties of \( M(t, c; x; w) \) which do not depend explicitly on \( c \) can be generalized to properties of \( L(s, t; x) \).

\[ M_t(f, P) = \left[ \int f^t(u) P(u) du \right]^{1/t} \]

In this generalization, \( s \) and \( t \) each play roles analogous to that of \( t \) in \( M(t, c; x; w) \).

2. Elementary properties of \( L(s, t; x) \). If \( P(u) \) is such that \( L'(s, t; x) \) is an improper integral, it is easily shown to converge uniformly in \( s, t, \) and \( x \) for \( 0 < m \leq x_i \leq M, 1 \leq i \leq n, \) all real \( s, \) and \( -T \leq t \leq T \). As a result of this uniform convergence and the continuity of \( M_t(x, u) \) in \( s, t, x, \) and \( u, L(s, t; x) \) is continuous in \( s, t, \) and \( x \).

In considering some limiting values of \( L(s, t; x) \), we use the notation \( x_{\text{max}} = \max\{x_1, x_2, \ldots, x_n\} \) and \( x_{\text{min}} = \min\{x_1, x_2, \ldots, x_n\} \).

**Theorem 1.**

(a) \( L(s, 0; x) = \lim_{t \to 0} L(s, t; x) = \exp \left[ \int_E \ln M_s(x, u) P(u) du \right] \),

(b) \( \lim_{t \to -\infty} L(s, t; x) = x_{\text{max}}, \) \( \lim_{t \to -\infty} L(s, t; x) = x_{\text{min}}, \)

(c) \( \lim_{t \to -\infty} L(s, t; x) = x_{\text{min}}. \)

**Proof.** Part (a) is an application of L'Hôpital's rule, differentiation with respect to \( t \) under the integral sign being permissible because the integral of the derivative converges uniformly for \( -T \leq t \leq T \) and \( 0 < m \leq x_i \leq M, 1 \leq i \leq n \). Parts (b) and (c) follow from properties of the integral mean \( M_t(f, P) [5, \text{p. 143}] \) with \( f(u) = M_s(x, u) \). Parts (d) and (e) follow from properties of \( M_s(x, u) [5, \text{Theorem 4}] \).

**Theorem 2.**

(a) \( L(s, t; x) \) is a strictly increasing function of \( x; \) i.e. if \( x_i \leq y_i \) for all \( i \) and \( x_j < y_j \) for some \( j \), then \( L(s, t; x) < L(s, t; y) \).

(b) If \( x_{\text{max}} > x_{\text{min}}, \) then \( L(s, t; x) \) is a strictly increasing function of \( t \).

(c) If \( x_{\text{max}} > x_{\text{min}}, \) then \( L(s, t; x) \) is a strictly increasing function of \( s \).

**Proof.** (a) From the definition of \( M_s(x, u) \) we see that \( M_s(x, u) \)
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<Ms(y, u). The result is evident by inspection of (1.2) and Theorem 1(a). Part (b) is a property of the integral mean Mt(f, P) [5, p. 144] with f(u) = Ms(x, u). For (c) it suffices to observe that f = Ms(x, u) is a strictly increasing function of s [5, Theorem 16] and that Mt(g, P) > Mt(f, P) if g(u) > f(u) for all u.

The following theorem shows that the elementary mean Ms(x, w) is a special case of the \( L(s, t; x) \) mean; the weights \( w = (w_1, w_2, \cdots, w_n) \) are the "natural weights" associated with the weight function \( P(u) \).

We define the natural weights by

\[
(2.1) \quad w_i = \int_E u_i P(u) du', \quad 1 \leq i \leq n.
\]

**Theorem 3.** If \( w \) denotes the natural weights, then \( L(s, s; x) = Ms(x, w) \).

**Proof.** If \( s \neq 0 \),

\[
L(s, s; x) = \left[ \int_E M_s(x, u) P(u) du' \right]^{1/s} = \left[ \int_E \sum_{i=1}^n u_i x_i^s P(u) du' \right]^{1/s}
= \left[ \sum_{i=1}^n x_i^s \int_E u_i P(u) du' \right]^{1/s} = Ms(x, w).
\]

If \( s = 0 \),

\[
L(0, 0; x) = \exp \left[ \int_E \ln \prod_{i=1}^n x_i^{u_i} P(u) du' \right]
= \exp \left[ \sum_{i=1}^n \ln x_i \int_E u_i P(u) du' \right]
= \exp \left[ \ln \prod_{i=1}^n x_i^{w_i} \right] = M_0(x, w).
\]

A given set of weights \( w \) occurs as the natural weights associated with a large class of functions \( P \). This class contains the family \( P(cw; u) [4, Equation (2.2)] \) where the natural weights are just the parameters \( w \). If \( P(u) = P(cw; u) \), \( L(s, t; x) \) becomes the "generalized hypergeometric mean" \( L(s, t; x; w) \). In particular, \( L(1, t, c; x; w) = M(t, c; x; w) \). For \( s \neq 0 \), \( L(s, t, c; x; w) \) can be expressed in terms of \( M(t, c; x; w) \) by using the identity

\[
(2.2) \quad L^r(s, t; x) = \left[ \int_E \left[ \sum_{i=1}^n u_i(x_i)^{s/r} \right]^{(t/r)/(s/r)} P(u) du' \right]^{r/t}
= L(s/r, t/r; x^r), \quad r, s, t \neq 0.
\]
A similar argument gives the same identity if $s$ or $t$ is zero. Putting $r = s$ and $P(u) = P(cw; u')$, we have

$$L(s, t; c; x; w) = [M(t/s, c; x^t; w)]^{1/s}.$$ 

Although Theorems 1 and 2 show at once that $x_{\text{min}} < L(s, t; x) < x_{\text{max}}$, the introduction of natural weights allows us to give sharper inequalities:

**Corollary 1.** Let $w$ denote the natural weights. If $x_{\text{max}} > x_{\text{min}}$ and $s < t$, then $M_*(x, w) < L(s, t; x) < M_*(x, w)$. The inequalities are reversed if $s > t$.

**Proof.** By Theorem 2, $L(s, t; x)$ is an increasing function of each of the parameters $s$ and $t$. Hence, applying Theorem 3, if $s < t$,

$$M_*(x, w) = L(s, s; x) < L(s, t; x) < L(t, t; x) = M_*(x, w),$$

with reversed inequalities if $s > t$.

It is well known [1, p. 9] that if $\log f(r, u)$ is convex in $r$, then $\log \int f(r, u)du$ is convex in $r$. To study the convexity of $L^*(s, t; x)$ in $s$, we need the analogous theorem that if $r \log f(r, u)$ is convex in $r$, then $r \log \int f(r, u)du$ is convex in $r$ for $r > 0$.

**Lemma 1.** Let $f^*(r, u)$ be continuous in $r$ and $u$ and log convex in $r$. Then, if $[\int f(r, u)P(u)du]^r$ is continuous, it is log convex in $r$ for $r > 0$.

**Proof.** If $f^*(r, u)$ is continuous and log convex in $r$, then $[f^*(r, u)]^{1/r} = f(r, u)$ is continuous and log convex in the variable $1/r$ for $r > 0$ [5, Theorem 119]. Since $f(r, u)$ is continuous in $u$ and log convex in $1/r$, $\int f(r, u)P(u)du$ is log convex in $1/r$ [1, p. 9]. Hence $[\int f(r, u)P(u)du]^r$ is log convex in $r$ for $r > 0$ [5, Theorem 119].

**Theorem 4.** (a) $L^*(s, t; x)$ is log convex in $s$ for $t/s \geq 0$. (b) $L^*(s, t; x)$ is log convex in $t$.

**Proof.** (a) For any fixed $t \neq 0$, $L^*(s, t; x) = [\int_M M_r^*(x, u)P(u)du']^{1/t} = [\int_M M_r(y, u)P(u)du']^{1/t}$, where $y_i = x_i$, $1 \leq i \leq n$, and $r = s/t$. Since $M_r^*(y, u)$ is log convex in $r$ [5, Theorem 87], Lemma 1 implies $[\int_M M_r(y, u)P(u)du']^r$ is log convex in $r$ for $r = s/t > 0$. But if a function is log convex in $r$, it is log convex in $tr = s$ for any fixed $t \neq 0$, [1, Theorem 1.10].

For $t = 0$, $L^*(s, 0; x) = \int_M M_r(x, u)P(u)du'$ is convex in $s$ since $M_r^*(x, u)$ is log convex in $s$.

(b) $L^*(s, t; x)$ is log convex in $t$ since $M_r^*(f, P)$ is log convex in $t$ [5, Theorem 197], and $L^*(s, t; x)$ has this form with $f(u) = M_*(x, u)$.

3. **Inequalities for $L(s, t; x)$.** The next two theorems are results of
properties of the mean \( M_s(x, u) \) and the integral mean \( M_t(f, P) \). The first comes from Theorems 24, 186 and 198 of [5]. The second comes from Theorems 12 and 188 of [5], with the special case \( t = 0 \) as an elementary result of properties of the logarithm. We use the notation \( x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \) and \( xy = (x_1y_1, x_2y_2, \cdots, x_ny_n) \).

**Theorem 5 (Minkowski).** Let \( x \) and \( y \) be vectors with \( x_i > 0 \) and \( y_i > 0 \), \( 1 \leq i \leq n \). Then, unless \( s = t = 1 \) or \( x_i = ky_i \), \( 1 \leq i \leq n \),

\[
L(s, t; x + y) < L(s, t; x) + L(s, t; y), \quad (s, t \geq 1),
\]

with reversed inequality if \( s, t \leq 1 \). Equality holds in the exceptional cases.

**Theorem 6 (Hölder).** Let \( x \) and \( y \) be vectors with \( x_i > 0 \) and \( y_i > 0 \), \( 1 \leq i \leq n \), and let \( p \) and \( q \) be real numbers greater than unity such that \( 1/p + 1/q = 1 \). Then, unless \( s = t = 0 \) or \( x_i^p = ky_i^p \), \( 1 \leq i \leq n \),

\[
L(s, t; xy) < L^{1/p}(s, t; x^p)L^{1/q}(s, t; y^q), \quad (s, t \geq 0),
\]

with reversed inequality if \( s, t \leq 0 \). Equality holds in the exceptional cases.

By defining the mean of a matrix of values \( x_{ij} \), both of the preceding theorems can be included in an analogue of the Jessen-Ingham inequality [5, Theorems 26 and 203]. The proof [7, p. 20] relies primarily on the Minkowski inequality and uses Hölder's inequality for a special case.

Finally, we shall show that \( L(s, t; x) \) satisfies a Kantorovich inequality [2, p. 208]. The proof proceeds by adapting a method due to Rennie [6, p. 982].

**Theorem 7 (Rennie).** Let \( 0 \leq x_i \leq B \), \( 1 \leq i \leq n \). Then if \( t \neq 0 \),

\[
L(t, s; x) + A'B^tL^{-t}(s, -t; x) = L^t(s, t; x) + A'B^tL^t(-s, t; 1/x) \leq A^t + B^t,
\]

with equality if and only if \( x_i = A \) or \( x_i = B \), \( 1 \leq i \leq n \).

**Proof.** The equality between the first and second members is seen by (2.2) with \( r = -1 \). To obtain the inequality we notice that \( M_s^t(x, u) \) is bounded between \( A^t \) and \( B^t \) and, following Rennie, we consider

\[
[M_s^t(x, u) - A^t][1 - B^tM_s^t(x, u)]P(u) \leq 0, \quad (t \neq 0).
\]

Integrating and rearranging, we have

\[
\int_B M_s^t(x, u)P(u)du' + A'B^t \int M_s^{-t}(x, u)P(u)du' \leq A^t + B^t.
\]
Equality holds if and only if $M'_i(x, u) = A^i$ or $M'_i(x, u) = B^i$ for all $u \in E$.

**Theorem 8 (Kantorovich).** If $t > 0$ and $0 < A \leq x_i \leq B$, $1 \leq i \leq n$, then

$$1 \leq L(s, t; x)/L(s, -t; x) = L(-s, t; x)L(s, t; 1/x) \leq \left[ (A^i + B^i)/2 \right]^{1/t} \left[ (A^{-t} + B^{-t})/2 \right]^{1/t},$$

with equality on the left if and only if $x_{\text{max}} = x_{\text{min}}$, and equality on the right if and only if $A = B$.

**Proof.** The left inequality holds because $L(s, t; x)$ is a strictly increasing function of $t$ by Theorem 2(b), unless $x_{\text{max}} = x_{\text{min}}$. The equality between the second and third members is due to (2.2) with $r = -1$. To obtain the right-hand inequality we start with Rennie's inequality (Theorem 7),

$$L'(s, t; x) + A'B'L^{-t}(s, -t; x) \leq A^t + B^t.$$  

Dividing by 2, applying the inequality of the arithmetic and geometric means to the left side, and squaring, we find

$$[L(s, t; x)/L(s, -t; x)]^t \leq (A^t + B^t)^2/4A'B^t$$

$$= \left[ (A^t + B^t)/2 \right] \left[ (A^{-t} + B^{-t})/2 \right].$$

Taking the $t$th root gives the desired inequality. If $A = B$, we have equality at each step of the proof; if $A \neq B$, then the conditions for equality in Theorem 7 imply strict inequality of the arithmetic and geometric means.

**References**


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