ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS
NEAR AN IRREGULAR SINGULARITY

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1. Introduction. In [2], the asymptotic behavior of solutions of
n-th-order homogeneous linear ordinary differential equations near a
singular point at \( \infty \), was investigated. The class of equations treated
in that paper (as well as here) roughly consists of those linear equa-
tions whose coefficients are complex functions defined and analytic
in unbounded sectorial regions (see \( \S 2(a) \) below), and have asymp-
totic expansions as \( x \to \infty \) in terms of real (but not necessarily inte-
gral) powers of \( x \) and/or functions which are of smaller rate of growth
(\( \ll \)) than all powers of \( x \) as \( x \to \infty \). (We are using here the concept of
asymptotic equivalence (\( \sim \)) as \( x \to \infty \), and the order relation "\(<\),"
introduced in [5, \( \S 13 \)].) However, it should be noted (see [5, \( \S 128(g) \)])
that the class of equations treated here includes, as a special case,
equations where no requirement is imposed except that each coeffi-
cient be analytic and have an asymptotic expansion (in the custom-
ary sense) of the form \( \sum c_j x^{-\lambda_j} \) with \( \lambda_j \) real and \( \lambda_j \to +\infty \) as \( j \to \infty \).
(For a summary of the necessary definitions from [5], and the needed
results from [2], see \( \S \S 2, 3 \) below.) In [2], solutions were sought
which were \( \sim \) to complex logarithmic monomials (i.e. functions of
the form, \( K x^{\alpha_0} (\log x)^{\alpha_1} (\log \log x)^{\alpha_2} \cdots (\log_q x)^{\alpha_q} \) for complex \( \alpha_j \) and
\( K \) with \( K \neq 0 \)). Associated with a linear equation in the class described
above, is an algebraic equation of degree at most \( n \) (see [1, \( \S 17 \)] or
[2, \( \S 3(e) \)]), which is a generalization of the classical "indicial equa-
tion at \( \infty \)." In [2, \( \S \S 5, 11 \)], it was shown that if \( p \) is the degree of this
corresponding algebraic equation, then there are precisely \( p \) complex
logarithmic monomials of the form \( M_j = x^{\beta_j} (\log x)^{\gamma_j} \) \((j = 1, \cdots, p)\),
where \( M_i \) is not \( \sim M_j \) if \( i \neq j \), such that any solution of the differential
equation which is \( \sim \) to a complex logarithmic monomial is \( \sim \) to a
constant multiple of some \( M_j \), and there exist solutions \( g_i \sim M_i \) such
that \( \{g_1, \cdots, g_p\} \) is a linearly independent set. When \( p = n \) (which
is a generalization of the notion of regular singularity at \( \infty \)), this
result is an asymptotic analog of one part of the classical Fuchs Regu-
arity Theorem. (See [4, p. 365] for a complete discussion of Fuchs' Theorem.)

When \( p < n \) (which generalizes the situation of an irregular singu-
laritity at \( \infty \)), a natural question is raised; namely, what is the

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asymptotic behavior of the remaining \( n-p \) solutions in a fundamental set? In this paper we answer this question in the case \( p = n-1 \), by proving (see §4 below) that when \( p = n-1 \), there is a solution \( g^* \), such that \( \{g_1, \ldots, g_{n-1}, g^*\} \) is a fundamental set, and such that \( g^* \) is either of smaller rate of growth as \( x \to \infty \) than all powers of \( x \), or is of larger rate of growth as \( x \to \infty \) than all powers of \( x \). This result is proved using the classical technique of reduction of order as well as results in [3] and [6] concerning the asymptotic behavior of solutions of certain first-order linear equations.

2. Concepts from [5]. (a) [5, §94]: Let \( a \) and \( b \) be real numbers, \( -\pi \leq a < b \leq \pi \). For each nonnegative real-valued function \( g \), defined on \( (0, (b-a)/2) \), let \( V(g) \) be the union (over \( \delta \in (0, (b-a)/2) \)) of all sectors, \( a + \delta < \arg(x - h(\delta)) < b - \delta \), where \( h(\delta) = g(\delta) \exp(i(a+b)/2) \).

The set of all \( V(g) \) (for all choices of \( g \)) is denoted \( F(a, b) \), and is a filter base which converges to \( \infty \). A statement is said to hold except in finitely many directions in \( F(a, b) \) (briefly, e.f.d. in \( F(a, b) \)) if there are finitely many points \( r_1 < r_2 < \ldots < r_q \) in \( (a, b) \), such that the statement holds in each of \( F(a, r_1), F(r_1, r_2), \ldots, F(r_q, b) \) separately.

(b) [5, §13]: If \( f \) is analytic in some \( V(g) \), then \( f \to 0 \) in \( F(a, b) \) means that for any \( \epsilon > 0 \) there is a \( g \), such that \( |f(x)| < \epsilon \) for all \( x \in V(g) \). The statement \( f \prec 1 \) in \( F(a, b) \) means that in addition to \( f \to 0 \), all functions \( \theta_j^k f \to 0 \) where \( \theta_j f = x \log x \cdots \log x f' \). The statements \( f_1 \prec f_2 \) and \( f_1 \sim f_2 \) mean respectively, \( f_1/f_2 < 1 \), \( f_1 - f_2 < f_2 \) and \( f_1 \sim c f_2 \) for some constant \( c \neq 0 \). An important property of the order relation \( \prec \), (proved in [5, §28]) is that \( f \prec 1 \) implies \( \theta_j f \prec 1 \) for all \( j \), so in particular, \( x f' \prec 1 \).

(c) [5, §49]: A logarithmic domain of rank zero, (briefly an \( \text{LD}_0 \)) over \( F(a, b) \), is a complex vector space \( E \) of functions (each analytic in some \( V(g) \)), which contains the constants, and such that any finite linear combination of elements of \( E \), with coefficients which for some \( q \geq 0 \) are functions of the form \( c x^{a_0} (\log x)^{a_1} \cdots (\log x)^{a_q} \) (for real \( a_j \)), is either \( \sim \) to a function of this latter form or is trivial (i.e. \( \prec x^a \) for all real \( a \)).

3. A result from [2]. Let \( \Omega(y) \) be an \( n \)th-order linear differential polynomial with coefficients in an \( \text{LD}_0 \) over \( F(a, b) \). If \( \theta \) is the operator \( \theta y = xy' \), \( \Omega(y) \) may be written \( \Omega(y) = \sum_{j=0}^n B_j(x) \theta^j y \), where the functions \( B_j \) belong to an \( \text{LD}_0 \). We assume \( B_n \) is nontrivial. By dividing through by the highest power of \( x \) which is \( \sim \) to a coefficient, \( B_j \), we may assume there is an integer \( p \geq 0 \) such that \( B_p \sim 1 \), \( B_j \prec 1 \) or \( \approx 1 \) for each \( j \), while for \( j > p \), \( B_j \prec x^{-\delta} \) for some \( \delta > 0 \). It is proved
in [2, §11] (using results obtained in [1] and [7]) that there exist \( p \) complex logarithmic monomials \( M_1, \ldots, M_p \), where each is of the form \( x^\alpha (\log x)^\beta \), and \( M_i \neq M_j \) if \( i \neq j \), such that any solution of \( \Omega(y) = 0 \) which is \( \sim \) to a complex logarithmic monomial is \( \approx M_i \) for some \( i \), and such that e.f.d. in \( F(a, b) \), the equation \( \Omega(y) = 0 \) possesses solutions \( g_1, \ldots, g_p \) with \( g_i \sim M_i \). If \( I \) is an interval such that \( g_1, \ldots, g_p \) exist in \( F(I) \), we say \( \{g_1, \ldots, g_p\} \) is a complete logarithmic set of solutions in \( F(I) \). The functions \( g_1, \ldots, g_p \) have the property that if for some constants \( c_1, \ldots, c_p \) the function \( \sum_{i=1}^{p} c_i g_i \) is trivial then all the \( c_i \) are 0.

4. The main theorem here. We will prove the following

**Theorem.** Let \( \Omega(y) = \sum_{i=0}^{n} B_i(x)\theta^i y \) be a linear differential polynomial where the functions \( B_i \) belong to an \( LD_0 \) over \( F(a, b) \), and where \( B_{n-1} \approx 1, B_j < 1 \) for all \( j \), \( B_n < x^{-\delta} \) for some \( \delta > 0 \) and \( B_n \) is non-trivial. Then \( (-B_{n-1}/(xB_n)) \sim cx^{-1+i/\tau} \) for some constant \( c \neq 0 \) and \( \tau > 0 \).

(This follows from the asymptotic properties of \( B_i \), and the definition of \( LD_0 \).) Let \( f(\phi) = \cos (\delta \phi + \arg c) \) for \( -\pi < \phi < \pi \). Then if \( (a_1, b_1) \) is any subinterval of \( (a, b) \) on which \( f(\phi) < 0 \) (respectively \( f(\phi) > 0 \)), then e.f.d. in \( F(a_1, b_1) \) the equation \( \Omega(y) = 0 \) has a fundamental set of solutions \( \{g_1, \ldots, g_{n-1}, g^*\} \), where \( \{g_1, \ldots, g_{n-1}\} \) is a complete logarithmic set and \( g^* < x^\alpha \) for all \( \alpha \) (respectively \( g^* > x^\alpha \) for all \( \alpha \)).

The proof of this theorem will be based on a sequence of lemmas, and will be concluded in §12.

5. **Notation.** If \( H \) is a nonzero solution of an \( n \)-th-order homogeneous linear differential equation \( \Lambda(y) = 0 \), then under the change of dependent variable \( y = Hu, z = \theta u \) followed by division by \( H \), we obtain an \((n-1)\)st-order equation denoted \( (H; \Lambda)(z) = 0 \). By induction we denote \( (H_i; H_{i-1}, \ldots, H_1; \Lambda) \), if defined, by \( (H_i, H_{i-1}, \ldots, H_1; \Lambda) \), where if \( i = 0 \), the latter is taken to be \( \Lambda \) itself.

6. **Uniform hypothesis.** Let \( \Omega \) satisfy the hypothesis of §4. Let \( I \) be any interval such that a complete logarithmic set \( \{g_1, \ldots, g_{n-1}\} \) exists in \( F(I) \) (see §3). Define functions \( h_1, \ldots, h_{n-1} \) as follows:

\[
h_1 = g_1 \quad \text{and} \quad h_{i+1} = (\theta \circ h_i^* \circ \cdots \circ \theta \circ h_i^*)(g_i+1),
\]

where \( h^* \) is the operator \( h^*y = y/h \). Let \( \Lambda_i \) be the operator \( \Lambda_i(y) = (\theta \circ h_i^* \circ \cdots \circ \theta \circ h_i^*)(y) \).

It is proved in [2, §12(B)] that

(A) \( h_i \) is \( \sim \) to a complex logarithmic monomial (whence by simple computation, \( \theta h_i/h_i < 1 \) or \( \approx 1 \) for all \( i \) and \( j \)).
(B) \( h_i, h_{i-1}, \ldots, h_1; \Omega \) is defined for all \( i = 0, \ldots, n-1 \).
7. **Lemma.** Assume the hypothesis of §6. For each \(i\), let

\[
(h_{i}, \ldots, h_{1}; \Omega)(y) = \sum_{s=0}^{n-i} B_{i,s} \theta^{s} y.
\]

Then, for each \(i\),

(a) \(B_{i,n-i} = B_{n}\), and

(b) \(B_{i,n-i-1} \sim B_{n-1}\) in \(F(I)\).

**Proof.** By simple computation,

\[
B_{i+1,s} = \sum_{j=s+1}^{n-i} B_{ij} \binom{j}{s+1} (h_{i+1})^{-1}\theta^{j-(s+1)} h_{i+1}.
\]

For \(i = 0\), (a) and (b) are clear. We proceed by induction, and assume (a) and (b) for \(i\). By (1), \(B_{i+1,n-(i+1)} = B_{i,n-i}\) which is \(B_{n}\) by induction hypothesis, proving (a) for \(i+1\). (b) follows for \(i+1\), from (1) and the asymptotic relations, \(B_{i,n-i-1} \sim B_{n-1} \approx 1\), \(B_{i,n-i} = B_{n} \approx x^{-\delta}\) and \((h_{i+1})^{-1}\theta h_{i+1} \approx 1\) or \(\approx 1\) (by §6(A)).

8. **Lemma.** Assume the hypothesis of §6. Then:

(a) There exists an analytic function \(W \sim (-B_{n-1}/xB_{n})\) such that the functions \(z_{0} = \exp \int W\) (where \(\int W\) stands for any primitive of \(W\) in \(F(I)\)), are solutions of \((h_{n-1}, \ldots, h_{1}; \Omega)(z) = 0\).

(b) \(W \sim c x^{-1+t}\) for some constants \(t > 0\) and \(c \neq 0\).

(c) The function \(f(\phi) = \cos (t\phi + \arg c)\) has only finitely many zeros in \((-\pi, \pi)\).

**Proof.** (a) By Lemma 7, (for \(i = n-1\)), the equation \((h_{n-1}, \ldots, h_{1}; \Omega)(z) = 0\) is \(B_{n} \theta z + E z = 0\) (where \(E \sim B_{n-1}\)), and hence is equivalent to the equation \(z - (z'/W) = 0\) where \(W = -E/(xB_{n})\). Thus (a) follows immediately.

(b) and (c) are obvious.

9. **Lemma.** Assume the hypothesis of §6 and let \(W\) be as in Lemma 8. Then if \(g\) is any function such that \(A_{n-1}(g) = \exp \int W\), we have \(\Omega(g) = 0\).

**Proof.** From the definition of \((H; \Lambda)\) it is clear that,

(A) If \(z_{0} = (\theta \circ H_{f})(y_{0})\) is a solution of \((H; \Lambda)(z) = 0\), then \(y = y_{0}\) is a solution of \(\Lambda(y) = 0\).

Let \(g\) be a function such that \(A_{n-1}(g) = \exp \int W\). Thus \(\exp \int W = (\theta \circ h_{n delicate).
10. Lemma. Let $c$ and $t$ be constants, $c \neq 0$ and $t > 0$. Let $W$ be any function $\sim cx^{-1+t}$ in some $F(J)$, and let $f(\phi) = \cos (t\phi + \arg c)$ for $-\pi < \phi < \pi$. ($f(\phi)$ is called the indicial function for $W$, and was introduced in [6, §61].) Then:

(a) If $J_1$ is any subinterval of $J$ on which $f(\phi) < 0$ (respectively, $f(\phi) > 0$), then for all $\alpha$, $\exp \int W < x^\alpha$ (respectively, $\exp \int W > x^\alpha$) in $F(J_1)$.

(b) If $J_2$ is any subinterval of $J$ on which $f(\phi)$ is never zero, then for any function $H$, which in $F(J_2)$ is $\sim$ to a complex logarithmic monomial, the equation $\theta y = H \exp \int W$ possesses a solution of the form $y = G \exp JW$, where $G$ is $\sim$ to a complex logarithmic monomial in $F(J_2)$. (In fact $G \sim H/(xW)$.)

Proof. (a) $z_0 = \exp \int W$ is a solution of $z - (z'/W) = 0$. Assume $f(\phi) < 0$ on $J_1$. In this case it is proved in [3, p. 271], that any solution of $z - (z'/W) = 0$ is $< 1$ in $F(J_1)$ so $z_0 < 1$. We now show that $z_0 < x^\alpha$ for all real $\alpha$. Assume the contrary. Then the set $A$ of all real $\alpha$ for which $z_0 < x^\alpha$, is nonempty (since it contains $\alpha = 0$) and is bounded below. Letting $\beta$ be the greatest lower bound of $A$, we clearly have $z_0 < x^\beta$ for all $\epsilon > 0$. Thus $z_0' < x^{\beta+\epsilon-1}$ (§2(b)). But $z_0 = z_0'/W$ and $W \sim cx^{-1+t}$. Thus $z_0 < x^{\beta+\epsilon-1}$ for all $\epsilon > 0$. Taking $\epsilon = t/2$, we obtain $z_0 < x^\beta$ which contradicts the definition of $\beta$. Hence $z_0 < x^\alpha$ for all $\alpha$ when $f(\phi) < 0$.

If now $f(\phi) > 0$, let $V = -W$. Then $V \sim -cx^{-1+t}$, so the indicial function for $V$ is $-f(\phi)$ which is strictly negative. Hence by the previous case, $\exp \int V < x^\alpha$ for all real $\alpha$, so $\exp \int W > x^\alpha$ for all $\alpha$.

(b) Let $H$ be given, $H \sim M$, where $M$ is a complex logarithmic monomial. Under the change of dependent variable, $y = uH \exp \int W$, the equation $\theta y = H \exp \int W$ is equivalent to,

\[(1) \quad xHu' + x(HW + H')u = H.\]

Since $H \sim M$, $H'/H$ is $< x^{-1}$ or $\approx x^{-1}$ and so is $< W$ (see §2(b)). Thus $HW + H' \sim HW$ so in some element of $F(J_2)$, $HW + H'$ is nowhere zero. Thus (1) is equivalent to,

\[(2) \quad u - (u'/V) = - (xV)^{-1}\]

where $V = -(W + (H'/H))$. Hence $V \sim -W$. The indicial function for $V$ is $-f(\phi)$ and so is nowhere zero on $J_2$. It is proved in [3, p. 271] that for such a $V$, an equation of the form $u - (u'/V) = E$ (where $E < 1$) always possesses a solution $< 1$ in $F(J_2)$. Since $-(xV)^{-1} < 1$, there exists $u_0 < 1$ in $F(J_2)$ satisfying equation (2). Thus,

\[(3) \quad u_0 = (u_0'/V) - (xV)^{-1}.\]
Since \( u_0 < 1 \), \( u'_0 < x^{-1} \) (see §2(b)). Thus \( (u'_0/V) < (xV)^{-1} \) so by (3), \( u_0 \sim -(xV)^{-1} \) in \( F(J_2) \). Hence \( u_0H \sim -M/(xV) \) in \( F(J_2) \). Since \( y_0 = u_0H \exp \int W \) is a solution of \( \theta y = H \exp \int W \), this proves (b).

11. Lemma. Assume the hypothesis of §6 and let \( W \) be as in Lemma 8. Let \( f(\phi) \) be the indicial function of \( W \), and let \( J \) be a subinterval of \( I \) on which \( f(\phi) \) is never zero. Then in \( F(J) \), the equation \( \Omega(y) = 0 \) possesses a solution of the form \( y^* = H \exp \int W \), where \( H \) is \( \sim \) to a complex logarithmic monomial in \( F(J) \). Furthermore, if on \( J \), \( f(\phi) < 0 \) (respectively, \( f(\phi) > 0 \)), then for all real \( \alpha \), \( y^* < x^\alpha \) (respectively, \( y^* > x^\alpha \)) in \( F(J) \).

Proof. (In this proof, \( L_i \) and \( N_i \) will denote complex logarithmic monomials.) By Lemma 9, any function \( y^* \) such that \( \Lambda_{n-1}(y^*) = \exp \int W \), will be a solution of \( \Omega(y) = 0 \). We now construct such a \( y^* \) by successive integrations using Lemma 10(b). By Lemma 10(b), there exists \( G_1 \sim L_1 \) in \( F(J) \) such that \( z_1 = G_1 \exp \int W \) is a solution of \( \theta z = \exp \int W \). Let \( y_1 = h_{n-1} z_1 \). Thus, \( \theta \circ h_{n-1}(y_1) = \exp \int W \), and \( y_1 = H_1 \exp \int W \) where \( H_1 \sim \Omega_1 \) by §6(A). Again by Lemma 10(b), there exists \( G_2 \sim L_2 \) such that \( z_2 = G_2 \exp \int W \) is a solution of \( \theta z = y_1 \). Letting \( y_2 = h_{n-2} z_2 \), we have \( \theta \circ h_{n-1} \circ \theta \circ h_{n-2}(y_2) = \exp \int W \) and \( y_2 = H_2 \exp \int W \) where by §6(A), \( H_2 \sim \Omega_2 \). Continuing this way, using Lemma 10(b), we obtain two sequences of functions \( z_2, \ldots, z_{n-1} \) and \( y_2, \ldots, y_{n-1} \) such that for each \( j \), \( z_j = G_j \exp \int W \) is a solution of \( \theta z = y_{j-1} \), and \( G_j \sim L_j \), and where \( y_j = h_{n-j} z_j \). Thus clearly, \( y_j = H_j \exp \int W \) where \( H_j \sim \Omega_j \) by §6(A). Let \( y^* = y_{n-1} \) and \( H = H_{n-1} \). Then clearly \( y^* = H \exp \int W \), and by construction \( \Lambda_{n-1}(y^*) = \exp \int W \) so \( \Omega(y^*) = 0 \) by Lemma 9. Let \( N_{n-1} = K \alpha q (\log x)^r \cdot \ldots \cdot \). Since \( H \sim \Omega_{n-1} \), it is easily verified that if \( \alpha_1 = 1 + \Re(q) \) and \( \alpha_2 = -1 + \Re(q) \) then \( H < x^{\alpha_1} \) while \( H > x^{\alpha_2} \). Since \( y^* = H \exp \int W \), we have \( y^* < x^{\alpha_1} \exp \int W \) and \( y^* > x^{\alpha_2} \exp \int W \) in \( F(J) \). If \( f(\phi) < 0 \) then by Lemma 10(a), in \( F(J) \), \( \exp \int W > x^\alpha \) for all \( \alpha \). Hence clearly \( y^* < x^\alpha \) for all \( \alpha \). Similarly, if \( f(\phi) > 0 \), then by Lemma 10(a), \( \exp \int W > x^\alpha \) for all \( \alpha \), so clearly, \( y^* > x^\alpha \) for all \( \alpha \).

12. Conclusion of proof of theorem (§4). If \( W \) is as in Lemma 8, and if \( f(\phi) \) is the indicial function of \( W \), then by Lemma 8(c), \( f(\phi) \) has only finitely many zeros, \( r_1 < r_2 < \cdots < r_q \) in \( (a, b) \). Let \( I_1 \) be any of the intervals \( (a, r_1), (r_1, r_2), \ldots, (r_q, b) \). Then by §3, e.f.d. in \( F(I_1) \), there exists a complete logarithmic set of solutions of \( \Omega(y) = 0 \). Letting \( I \) be any subinterval of \( I_1 \), such that a complete logarithmic set of solutions \( \{g_1, \ldots, g_{n-1}\} \) exists on \( F(I) \), then in \( F(I) \), by Lemma 11, there is a solution \( g^* \) of \( \Omega(y) = 0 \), which is of the form
$g^* = H \exp \int W$ where $H \sim$ to a complex logarithmic monomial in $F(I)$. If $f(\phi) > 0$ on $I$, $g^* \sim x^\alpha$ for all $\alpha$, while if $f(\phi) < 0$ on $I$, $g^* \sim x^\alpha$ for all $\alpha$. Clearly the sets $\{g_1, \ldots, g_{n-1}, g^*\}$ exist e.f.d. in $F(I)$ (and e.f.d. in $F(a, b)$).

To conclude the proof of §4, we must show that $\{g_1, \ldots, g_{n-1}, g^*\}$ is a linearly independent set.

Suppose for some constants $C_1, \ldots, C_n$ we have $\sum_{i=1}^{n-1} C_i g_i + C_n g^* = 0$. If $g^* \sim x^\alpha$ for all $\alpha$, then clearly $-C_n g^*$ is also trivial. Thus since $\sum_{i=1}^{n-1} C_i g_i = -C_n g^*$, we have $C_1 = \cdots = C_{n-1} = 0$ (see end of §3). Hence $C_n g^* = 0$. Since $g^* = H \exp \int W$, $g^*$ is nonzero, so $C_n = 0$ also.

If $g^* \sim x^\alpha$ for all $\alpha$, then writing the dependence relation as $C_n = -\sum_{i=1}^{n-1} C_i (g_i / g^*)$, we see that the right side of this relation $\to 0$, so $C_n = 0$. Since $\{g_1, \ldots, g_{n-1}\}$ is linearly independent, $C_1 = \cdots = C_{n-1} = 0$ also. Hence in both cases $\{g_1, \ldots, g_{n-1}, g^*\}$ is a fundamental set.

**Bibliography**


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