CLUSTER SET THEOREMS FOR UNIFORMLY CONVERGENT SEQUENCES OF FUNCTIONS

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1. Introduction. Let \( f(z) \) be a complex-valued function defined in \( D: \{ |z| < 1 \} \), with values on the Riemann sphere \( S \). At any point \( e^{i\vartheta} \) of \( C: \{ |z| = 1 \} \), the (interior) cluster set, \( C_D(f, e^{i\vartheta}) \), is defined as follows: \( \alpha \in C_D(f, e^{i\vartheta}) \) if there exists a sequence \( \{ z_n \} \) in \( D \) such that \( \lim_{n \to \infty} z_n = e^{i\vartheta} \) while \( \lim_{n \to \infty} f(z_n) = \alpha \). For any point \( e^{i\vartheta} \) of \( C \), \( C_D(f, e^{i\vartheta}) \) is closed and nonempty. If \( G \) is a subset of \( D \) whose closure contains \( e^{i\vartheta} \), the partial cluster set, \( C_G(f, e^{i\vartheta}) \), is defined analogously by requiring the sequence \( \{ z_n \} \) to lie in \( G \). (For a more detailed introduction to the theory of cluster sets, see [1].)

In this paper we consider a sequence \( \{ f_n(z) \} \) of functions defined in \( D \) and converging uniformly to a function \( f(z) \) in \( D \). In §2 we consider the convergence of a sequence of cluster sets for \( \{ f_n(z) \} \) at \( e^{i\vartheta} \) to the corresponding cluster set for \( f(z) \) at \( e^{i\vartheta} \). The function \( f(z) \) is said to have an ambiguous point at \( e^{i\vartheta} \) if there exist two simple arcs, \( K \) and \( L \), in \( D \) terminating at \( e^{i\vartheta} \) for which \( C_K(f, e^{i\vartheta}) \cap C_L(f, e^{i\vartheta}) = \emptyset \). (The original references and statements about ambiguous points may be found in [1, p. 39].) In §3 we relate the ambiguous points of \( f(z) \) to those of \( \{ f_n(z) \} \). In §4 we present related results for mappings from an arbitrary topological space into a compact metric space.

Let \( Z \) be a compact metric space with metric \( d \). For any nonempty closed subset \( A \) of \( Z \) and any \( \varepsilon > 0 \), we let \( A + \varepsilon = \{ z \in Z: \exists a \in A \text{ with } d(a, z) < \varepsilon \} \). If \( \{ A_n \} \) is a sequence of nonempty closed subsets of \( Z \), we say \( \lim_{n \to \infty} A_n = A \) if for any \( \varepsilon > 0 \) there exists an integer \( N \) such that \( A_n \subseteq A + \varepsilon \) and \( A \subseteq A_n + \varepsilon \) whenever \( n > N \). It is in this sense that we discuss the convergence of a sequence of cluster sets.

2. Our hypothesis in the following is that \( \{ f_n(z) \} \) is a sequence of arbitrary complex-valued functions converging uniformly in \( D \) to a function \( f(z) \). For simplicity, we shall denote by \( |a - b| \) the distance between \( a \) and \( b \) on \( S \) in the spherical metric.

**Theorem 1.** If \( e^{i\vartheta} \) is any point of \( C \), then \( \lim_{n \to \infty} C_D(f_n, e^{i\vartheta}) = C_D(f, e^{i\vartheta}) \), and this convergence is uniform in \( e^{i\vartheta} \).

**Proof.** Let \( e^{i\vartheta} \) be an arbitrary point of \( C \), and let \( \varepsilon > 0 \) be arbitrarily chosen. Then for some positive integer \( N \), whenever \( n > N \) and \( z \in D \),

\[
|f_n(z) - f(z)| < \varepsilon/3.
\]

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Choose any \( n > N \), and select any point \( \alpha_n \) from \( C_D(f_n, e^{i\theta}) \). Then there exists a sequence \( \{a_k(n)\} \subset D \) such that \( \lim_{k \to \infty} a_k(n) = e^{i\theta} \) and \( \lim_{n \to \infty} f_n(a_k(n)) = \alpha_n \). For some subsequence \( \{a_j(n)\} \) of \( \{a_k(n)\} \) there exists \( \lim_{j \to \infty} f(a_j(n)) = w \), where \( w \in C_D(f, e^{i\theta}) \), while \( \lim_{n \to \infty} f_n(a_j(n)) = \alpha_n \). Now we can find an integer \( J \) such that whenever \( j > J \), both \( |f(a_j(n)) - w| < \epsilon/3 \) and \( |f_n(a_j(n)) - \alpha_n| < \epsilon/3 \). But then \( |\alpha_n - w| \leq |\alpha_n - f(a_j(n))| + |f_n(a_j(n)) - f(a_j(n))| + |f(a_j(n)) - w| < \epsilon \) whenever \( j > J \), so that for \( \alpha_n \in C_D(f_n, e^{i\theta}) \), \( n > N \), we have \( \alpha_n \in C_D(f, e^{i\theta}) + \epsilon \).

Now let any \( w \in C_D(f, e^{i\theta}) \) be chosen. Then there exists a sequence \( \{a_k\} \subset D \) with \( \lim_{k \to \infty} a_k = e^{i\theta} \) and \( \lim_{k \to \infty} f(a_k) = w \). For each \( n > N \) we can select a subsequence \( \{a_k(n)\} \) of \( \{a_k\} \) along which \( f_n(z) \) has a limit \( \alpha_n \in C_D(f_n, e^{i\theta}) \), while \( \lim_{k \to \infty} f(a_k(n)) = w \). A repetition of the argument above will show that if \( w \in C_D(f, e^{i\theta}) \), then \( w \in C_D(f_n, e^{i\theta}) + \epsilon \) for each \( n > N \).

By our definition, \( \lim_{n \to \infty} C_D(f_n, e^{i\theta}) = C_D(f, e^{i\theta}) \), and since \( N \) above is independent of the point \( e^{i\theta} \), this convergence is uniform in \( e^{i\theta} \).

An obvious modification of the proof of Theorem 1 yields

**Theorem 2.** Let \( G \) be any subset of \( D \) whose closure contains a point \( e^{i\theta} \) of \( C \). Then \( \lim_{n \to \infty} C_G(f_n, e^{i\theta}) = C_G(f, e^{i\theta}) \). In particular, if \( G \) is the radius, \( \rho \), to \( e^{i\theta} \), \( \lim_{n \to \infty} C_\rho(f_n, e^{i\theta}) = C_\rho(f, e^{i\theta}) \).

From Theorem 2 we may state the following

**Corollary.** If \( L \) is any simple arc in \( D \) terminating at \( e^{i\theta} \) on \( C \), then \( f(z) \) has a limit \( \gamma \) as \( z \) approaches \( e^{i\theta} \) along \( L \) if, and only if, \( \lim_{n \to \infty} C_L(f_n, e^{i\theta}) = \{ \gamma \} \). In particular, \( f(z) \) has a radial limit \( \lim_{r \to 1} f(re^{i\theta}) = \gamma \) if, and only if, \( \lim_{n \to \infty} C_\rho(f_n, e^{i\theta}) = \{ \gamma \} \).

A point \( \alpha \) belongs to the boundary cluster set, \( C_B(f, e^{i\theta}) \), for a function \( f(z) \) at \( e^{i\theta} \) if there exist: (i) a sequence \( \{\tau_k\} \) of points on \( C - \{e^{i\theta}\} \) such that \( \lim_{k \to \infty} \tau_k = e^{i\theta} \); and (ii) a sequence of points \( \{\omega_k\} \) with \( \omega_k \in C_D(f, \tau_k) \) such that \( \lim_{k \to \infty} \omega_k = \alpha \). If in (ii) we require that \( \omega_k \in C_\rho(f, \tau_k) \), the radial cluster set of \( f(z) \) at \( \tau_k \), then we have the definition of the radial boundary cluster set, \( C_{BR}(f, e^{i\theta}) \), for \( f(z) \) at \( e^{i\theta} \). (For details of the role these cluster sets play in the boundary behavior of functions meromorphic in \( D \), see [1] and the paper of W. B. Woolf [3].)

**Theorem 3.** For any point \( e^{i\theta} \) on \( C \), \( \lim_{n \to \infty} C_B(f_n, e^{i\theta}) = C_B(f, e^{i\theta}) \), and \( \lim_{n \to \infty} C_{BR}(f_n, e^{i\theta}) = C_{BR}(f, e^{i\theta}) \).

**Proof.** It suffices to prove the first of these statements. Let any \( \epsilon > 0 \) be given; for some integer \( N \), \( |f_n(z) - f(z)| < \epsilon/9 \) for all \( z \) in \( D \) whenever \( n > N \). Choose any integer \( n > N \), and let \( \alpha(n) \) be an arbi-
trary point of \(C_B(f_n, e^{i\theta})\). Then there exists a sequence \(\{\tau_k(n)\}\) on \(C - \{e^{i\theta}\}\) and a sequence \(\{\omega_k(n)\}\) such that \(\lim_{k \to \infty} \tau_k(n) = e^{i\theta}, \omega_k(n) \in C_D(f_n, \tau_k)\), and \(\lim_{k \to \infty} \omega_k(n) = \alpha(n)\).

From the proof of Theorem 1 we have \(C_D(f_n, \tau) \subset C_D(f, \tau) + \epsilon/3\) for \(n > N\) and any \(\tau \in C\); thus for each value of \(k\) there exists a point \(\gamma_k(n) \in C_D(f, \tau_k)\) such that \(|\omega_k(n) - \gamma_k(n)| < \epsilon/3\). From the sequence \(\{\gamma_k(n)\}\) we can select a convergent subsequence—which for simplicity we denote by \(\{\gamma_k(n)\}\) itself—with a limit \(\gamma \in C_B(f, e^{i\theta})\). There exists an integer \(K(\epsilon, n)\) such that \(|\omega_k(n) - \alpha(n)| < \epsilon/3\) and \(|\gamma_k(n) - \gamma| < \epsilon/3\) whenever \(k > K(\epsilon, n)\). Then for any \(k > K(\epsilon, n)\) we may write \(|\alpha(n) - \gamma| \leq |\alpha(n) - \omega_k(n)| + |\omega_k(n) - \gamma_k(n)| + |\gamma_k(n) - \gamma| < \epsilon\), so that \(\alpha(n) \in C_B(f, e^{i\theta}) + \epsilon\), or \(C_B(f_n, e^{i\theta}) \subset C_B(f, e^{i\theta}) + \epsilon\) for \(n > N\).

We wish to show also that for \(n > N\) \(C_B(f, e^{i\theta}) \subset C_B(f_n, e^{i\theta}) + \epsilon\). Let \(\alpha\) be an arbitrary point of \(C_B(f, e^{i\theta})\). Then for a sequence \(\{\tau_k\}\) on \(C - \{e^{i\theta}\}\) with \(\lim_{k \to \infty} \tau_k = e^{i\theta}\) there is a sequence \(\{\omega_k\}\) such that \(\omega_k \in C_D(f, \tau_k)\) and \(\lim_{k \to \infty} \omega_k = \alpha\).

For \(n > N\) and any \(\tau \in C\) we have \(C_D(f, \tau) \subset C_D(f_n, \tau) + \epsilon/3\). Thus for fixed \(n > N\) and each \(k\) we may select a point \(\gamma_k(n) \in C_D(f_n, \tau_k)\) such that \(|\omega_k - \gamma_k(n)| < \epsilon/3\). From this point on the argument repeats that above to show that \(C_B(f, e^{i\theta}) \subset C_B(f_n, e^{i\theta}) + \epsilon\) for \(n > N\). Hence \(\lim_{n \to \infty} C_B(f_n, e^{i\theta}) = C_B(f, e^{i\theta})\) for any point \(e^{i\theta}\) on \(C\).

Similar statements of convergence can be made in terms of other types of cluster sets at a point on \(C\).

3. Ambiguous points. If \(\{f_n(z)\}\) converges uniformly in \(D\) to \(f(z)\), it is an easy consequence of Theorem 2 that each ambiguous point of \(f(z)\) on \(C\) is an ambiguous point for all but finitely many functions \(f_n(z)\). Thus a function defined in \(D\) having an ambiguous point on \(C\) cannot be uniformly approximated in \(D\) by functions having no ambiguous points.

However, the limit of a uniformly convergent sequence of functions, each with an ambiguous point on \(C\), need not have an ambiguous point. As a simple example, define a sequence \(\{f_n(z)\}\), where \(f_n(z) = z\) for \(z \in D - K - L\), \(f_n(z) = e^{i\theta} + 1/n\) for \(z \in K\), \(f_n(z) = e^{i\theta} - 1/n\) for \(z \in L\), where \(n = 1, 2, 3, \cdots\) and \(K, L\) are simple arcs in \(D\) terminating at \(e^{i\theta}\) on \(C\). The sequence converges uniformly in \(D\) to a function \(f(z)\), with \(f(z) = z\) for \(z \in D - K - L\), \(f(z) = e^{i\theta}\) for \(z \in K \cup L\). For each \(n\), \(e^{i\theta}\) is an ambiguous point of \(f_n(z)\), but \(f(z)\) has no ambiguous points.

If we assign a crude measure to the extent to which a point of \(C\) is ambiguous for a function in \(D\), we can relate the points which are "uniformly" ambiguous for the uniformly convergent sequence \(\{f_n(z)\}\) and the ambiguous points of their limit \(f(z)\). Let us say that
f(z) is \( \delta \)-ambiguous at \( e^{i\theta} \) if there exist simple arcs \( K \) and \( L \) in \( D \) terminating at \( e^{i\theta} \) such that for each \( \alpha \in C_K(f, \ e^{i\theta}) \) and each \( \beta \in C_L(f, \ e^{i\theta}) \), \( |\alpha - \beta| \geq \delta > 0 \).

**Theorem 4.** If for all \( n \) and some \( \delta > 0 \) \( f_n(z) \) is \( \delta \)-ambiguous at \( e^{i\theta} \), then \( f(z) \) is ambiguous at \( e^{i\theta} \).

**Proof.** Let \( \rho \) be chosen, \( 0 < \rho < \delta \), and let \( \epsilon \) be chosen, \( 0 < \epsilon < \delta - \rho \). For some integer \( N \) and all \( z \) in \( D \), \( |f_n(z) - f(z)| < \epsilon/4 \) when \( n > N \). Select any \( n > N \). For this \( n \) there exist simple arcs, \( K = K(n) \) and \( L = L(n) \), in \( D \) terminating at \( e^{i\theta} \) such that \( |\alpha - \beta| \leq 5 > 0 \) for all \( \alpha \in C_K(f, \ e^{i\theta}), \beta \in C_L(f, \ e^{i\theta}) \).

Let \( \alpha, \beta \) be arbitrarily chosen from \( C_K(f, \ e^{i\theta}), C_L(f, \ e^{i\theta}) \), respectively. We show that \( |\alpha - \beta| \geq \rho \). For some sequences \( \{a_j\} \subseteq K \), \( \{b_j\} \subseteq L \), we have \( \lim_{j \to \infty} f(a_j) = \alpha, \lim_{j \to \infty} f(b_j) = \beta \). From these sequences we can select subsequences \( \{a'_j\}, \{b'_j\} \) such that \( \lim_{j \to \infty} f_n(a'_j) = \gamma \in C_K(f_n, \ e^{i\theta}), \lim_{j \to \infty} f_n(b'_j) = \lambda \in C_L(f_n, \ e^{i\theta}) \), \( \lim_{j \to \infty} f_n(a'_j) = \gamma \in C_K(f_n, \ e^{i\theta}), \lim_{j \to \infty} f_n(b'_j) = \lambda \in C_L(f_n, \ e^{i\theta}) \).

We can find an integer \( J \) such that for \( J > J, |f_n(a'_j) - \gamma| < \epsilon/8, |f_n(b'_j) - \lambda| < \epsilon/8, |f(a'_j) - \alpha| < \epsilon/8, |f(b'_j) - \beta| < \epsilon/8 \). Then for \( J > J, |\alpha - \gamma| \leq |\alpha - f(a'_j)| + |f(a'_j) - f_n(a'_j)| + |f_n(a'_j) - \gamma| < \epsilon/2 \); and \( |\beta - \lambda| \leq |\beta - f(b'_j)| + |f(b'_j) - f_n(b'_j)| + |f_n(b'_j) - \lambda| < \epsilon/2 \). Now \( |\lambda - \gamma| \geq \delta \) by hypothesis, so \( \delta \leq |\lambda - \gamma| \leq |\lambda - \beta| + |\beta - \alpha| + |\alpha - \gamma| < \epsilon + |\alpha - \beta| < (\delta - \rho) + |\alpha - \beta| \), and \( |\alpha - \beta| > \rho \). Consequently, \( f(z) \) is ambiguous at \( e^{i\theta} \).

4. Let \( X \) be an arbitrary topological space and \( Z \) be a compact metric space with metric \( \rho \). For each \( x \in X \) denote by \( \mathcal{U}_x \) the collection of open sets in \( X \) containing \( x \). For any nonempty subset \( T \) of \( X \) let \( f \) be any mapping of \( T \) into \( Z \). J. D. Weston [2] defined the cluster set of \( f \) at a point \( t \in T \) to be \( C(f; \ t) = \bigcap \{ f(U) \mid U \in \mathcal{U}_t \} \), where \( \bigcap \) represents the intersection over all \( U \in \mathcal{U}_t \), and \( \mathcal{C}(A) \) denotes the closure of \( A \). For any \( f \) mapping \( T \) into \( Z \) and any \( t \in T \), we see that \( C(f; \ t) \) is nonempty and closed.

We state for reference the following lemma [2, p. 436].

**Lemma.** Let \( t \in T \) and \( K \) be a compact set in \( Z \). Suppose that, corresponding to each \( U \in \mathcal{U}_t \), a closed set \( F(U) \) in \( Z \) is prescribed so that:

(i) if \( U_1 \subset U_2 \), then \( F(U_1) \subset F(U_2) \); (ii) \( K \cap [\bigcap \mathcal{U}_t F(U)] = \emptyset \). Then there exists at least one \( U \in \mathcal{U}_t \) such that \( K \cap F(U) = \emptyset \).

Let \( \{f_n\} \) be a sequence of mappings of \( T \) into \( Z \) which converges uniformly on \( T \) to a mapping \( f \). That is, given any \( \epsilon > 0 \) there exists integer \( N \) such that whenever \( n > N, \rho [f_n(t), f(t)] < \epsilon \) for all \( t \in T \).
Theorem 5. For any point \( t \in T \), \( \lim_{n \to \infty} C(f_n; t) = C(f; t) \), and this convergence is uniform in \( t \).

Proof. Let \( \epsilon > 0 \) be given. Then for some integer \( N \), whenever \( n > N \), \( \rho[f_n(t), f(t)] < \epsilon/4 \) for all \( t \in T \). Choose any \( s \in T \); suppose for some integer \( n > N \) that there exists \( \alpha_n \in C(f_n; s) \) such that \( \rho(\alpha_n, \alpha) \geq \epsilon \) for all \( \alpha \in C(f; s) \).

Let \( K = \{ z \in Z : \rho(\alpha_n, z) \leq \epsilon/2 \} \); \( K \) is compact and \( K \cap C(f; s) = \emptyset \). Using the lemma with \( F(U) = \text{Cl}[f(U)] \), we have \( K \cap \text{Cl}[f(U)] = \emptyset \) for some \( U \in \mathcal{U} \). For each \( t \in U \), \( \rho[\alpha_n, f(t)] \geq \epsilon/2 \). But since \( \alpha_n \in C(f_n; s) \), \( \alpha_n \in \text{Cl}[f_n(U)] \), and we can find some \( t' \in U \) with \( f_n(t') \neq \alpha_n \) and \( \rho[\alpha_n, f_n(t')] < \epsilon/4 \). Now \( \epsilon/2 \leq \rho[\alpha_n, f(t')] \leq \rho[\alpha_n, f_n(t')] + \rho[f_n(t'), f(t')] < \epsilon/2 \). Thus for \( n > N \) and any \( \alpha_n \in C(f_n; s) \), there must be some \( \alpha \in C(f; s) \) such that \( \rho(\alpha_n, \alpha) < \epsilon \), and for \( n > N \) and any \( t \in T \) we have \( C(f_n; t) \subset C(f; t) + \epsilon \).

Now choose any \( s \in T \) and suppose there exists \( n > N \) and \( \alpha \in C(f; s) \) for which \( \rho(\alpha, \alpha_n) \geq \epsilon \) for any \( \alpha_n \in C(f_n; s) \). If \( K = \{ z \in Z : \rho(\alpha, z) \leq \epsilon/2 \} \) and \( F(U) = \text{Cl}[f(U)] \), then \( K \cap C(f_n; s) = \emptyset \), and the Lemma gives us a set \( U \in \mathcal{U} \), for which \( K \cap \text{Cl}[f(U)] = \emptyset \). Hence for each \( t \in U \), \( \rho[\alpha, f_n(t)] \geq \epsilon/2 \). Since \( \alpha \in C(f; s) \), we can find \( t' \in U \) with \( f(t') \neq \alpha \), \( \rho[f(t'), \alpha] < \epsilon/4 \), and again we have \( \epsilon/2 \leq \rho[f_n(t'), \alpha] \leq \rho[f_n(t'), f(t')] + \rho[f(t'), \alpha] < \epsilon/2 \). Thus for any \( n > N \) and any \( \alpha \in C(f; s) \), there exists \( \alpha_n \in C(f_n; s) \) such that \( \rho(\alpha, \alpha_n) < \epsilon \), so that for \( n > N \) and all \( t \in T \), \( C(f; t) \subset C(f_n; t) + \epsilon \).

Therefore, \( \lim_{n \to \infty} C(f_n; t) = C(f; t) \), and since \( N \) is independent of \( t \in T \), this limit is uniform in \( t \).

If \( t \) is a point of \( T \), let \( A \) be any subset of \( T \) such that \( t \in \text{Cl}(A) \). As a generalization of the boundary cluster set, Weston [2] defined the cluster set \( C^A(f; t) = \cap \{ \text{Cl}[M(f; U; A)] \} \), where for each \( U \in \mathcal{U} \), \( M(f; U; A) = \cup_{A \cap U} C(f; a) \). In addition, as a generalization of the partial cluster set, let \( C_A(f; t) = \cap \text{Cl} f(A \cap U) \). Then simple modifications in the proof of Theorem 5 will yield

Theorem 6. \( \lim_{n \to \infty} C^A(f_n; t) = C^A(f; t) \) and \( \lim_{n \to \infty} C_A(f_n; t) = C_A(f; t) \).

Bibliography


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