INTERPOLATION ON FINITE OPEN RIEMANN SURFACES

E. L. STOUT[1]

I. In a previous paper [5], we have considered the question of interpolation on finite open Riemann surfaces; it is our intention to show in this note that the problem can be treated in an essentially simpler way.

We shall suppose given a finite open Riemann surface, R. Such a surface is, by definition, the complement in some compact Riemann surface of a finite family of closed, pairwise disjoint discs each of which we may assume to have an analytic simple closed curve as boundary. A set $E \subset R$ is called an *interpolation set for R* if given a bounded, complex valued function α on E, there is a bounded holomorphic function f on R such that $f \mid E = \alpha$. It is convenient to denote by $H_\infty[R]$ the collection of all functions which are bounded and holomorphic on R.

For our characterization of interpolation sets in R we introduce a functional $d_R(z, E)$ as follows: If $z \in E$, then

$$d_R(z, E) = \sup\{\, |f(z)| : f \in H_\infty[R], f \mid (E - \{z\}) = 0, \|f\|_R \leq 1\}$$

where $\|f\|_R = \sup\{\, |f(w)| : w \in R\}$ is the usual supremum norm.

II. We can characterize the interpolation sets for R in terms of the functional d_R as follows.

THEOREM. *The set $E \subset R$ is an interpolation set for R if and only if* $\inf\{d_R(z, E): z \in E\} > 0$.

PROOF. Denote by Γ the boundary of R so that Γ consists of a finite collection of pairwise disjoint analytic simple closed curves. It is convenient to make use of an *Ahlfors map* for R, i.e., a function π continuous on $\overline{R}$ and holomorphic in R which is constantly of modulus one on Γ. The existence of such functions was established by Ahlfors in [1]; an alternative proof of their existence is in [4]. It is easy to see that $d\pi$, the differential of π, does not vanish on Γ, and thus π acts as a local homeomorphism at every point of Γ.

Let us suppose, initially, that E satisfies the following very stringent condition: We assume the existence of discs $V \subset \overline{V} \subset W$ in the ambient compact surface both of which meet Γ in open arcs and

which, moreover, have the properties that $E \subset V$ and π is one-to-one on $\overline{W}$. The hypothesis of the theorem implies that for each $z \in E$, $d_{W \cap R}(z, E) > \delta$ for some fixed $\delta > 0$ and all $z \in E$. Consequently, by a well-known theorem of Carleson [3, p. 203], the set E is an interpolation set for $W \cap R$. (For this formulation of Carleson's theorem, see [5, Lemma 5.1].) It follows then that the set $\pi[E]$ is an interpolation set for $\pi[W \cap R]$. Since $\pi[E] \subset \pi[V]$, and since π is one-to-one on $\overline{W}$, it follows from [5, Theorem 5.3] that $\pi[E]$ is an interpolation set for U, the open unit disc. But this clearly implies that E is an interpolation set for R.

We now show how this very special case of our theorem implies the general case. For each $p \in \Gamma$ there exist discs $V_p \subset \overline{V}_p \subset W_p$ as above such that $E \cap V_p$ is an interpolation set for R. Finitely many of the neighborhoods V_p, denote them by $V_1, \cdots, V_m$, cover Γ. Since the assumption of the theorem clearly implies that the set E is discrete, it follows that all but finitely many points of E lie in $\bigcup_{j=1}^{m} V_j$. The union of an interpolation set and a finite set is again an interpolation set whence it suffices for us to prove E to be an interpolation set for R under the assumption that $E \subset \bigcup_{j=1}^{m} V_j$. Let us write $E = E_1 \cup \cdots \cup E_m$ where this is a disjoint union and where $E_j \subset V_j$.

We shall prove the existence of functions $b_1, \cdots, b_m \in H_\infty[R]$ with the properties that $|b_j| > \epsilon$ on E_j and b_j vanishes on E_k if $k \neq j$. Here ϵ is some fixed positive number independent of j. Let us suppose the functions b_j have been constructed, and consider a bounded function α on E. Since E_j is an interpolation set for R and since b_j is bounded away from 0 on E_j, there exists $h_j \in H_\infty[R]$ such that $h_j(z) = \alpha(z)/b_j(z)$ for all $z \in E_j$. If $H = \sum_{j=1}^{m} b_j h_j$, then H is a bounded analytic function on R which agrees on E with α.

We conclude the proof of the sufficiency of our condition by establishing the existence of the functions b_j. In this proof we use the uniformization theorem. The universal covering surface of R is the open unit disc, U. Let $\Upsilon: U \to R$ be a uniformizing map, and let $\mathfrak{G}$ be the associated group of covering transformations. Denote by D the canonical fundamental domain for the group $\mathfrak{G}$ so that D is a certain open set together with a portion of its boundary which is mapped by Υ onto R in a one-to-one fashion. Denote by E_j' the set $\Upsilon^{-1}[E_j] \cap D$ and by E_j'' the set $\Upsilon^{-1}[E_j]$.

Let B_j be the Blaschke product with zero set, the set $\bigcup \{E_k'': k \neq j\}$ and with simple zeros. It is the case that if δ is a lower bound for the numbers $d_R(z, E)$ for z in E, then $|B_j(w)| > \delta$ for all w in E_j''. To establish this assertion, let $z \in E_j''$ be given. There is $f \in H_\infty[R]$ which is bounded by one, which is at least δ in modulus at the point $\Upsilon(z)$,

and which vanishes on the set $E - \{\Upsilon(z)\}$. Thus, the element $f \circ \Upsilon$ of $H_\infty[U]$ vanishes on $\bigcup\{E_k'' : k \neq j\}$, is at least δ in modulus at z, and is bounded by one. We can write $f \circ \Upsilon = BSF$ where B is a Blaschke product, S is a singular function, and F is an outer function. Since $\|f \circ \Upsilon\|_U \leq 1$, it follows that $\|F\|_U \leq 1$. Also, S is bounded by one. It follows therefore that $|B(z)|$ is at least δ. The zero set of B contains that of B_j, and since B_j has only simple zeros, we may write $B = B_j B'$ where B' is some other Blaschke product. But this obviously implies that $|B_j(z)|$ is at least δ in modulus. This calculation is independent of the choice of z in E_j'', so the assertion concerning B_j is established.

The function B_j is not invariant under the group $\mathfrak{G}$. It does, however, have the weaker property that $|B_j|$ is invariant. This is a consequence of the fact that if $[z, w] = |(z-w)/(1-z\bar{w})|$ and if g is a conformal mapping of the open unit disc onto itself, then $[g(z), g(w)] = [z, w]$. For the details, see $[6,$ p. $510]$. It is easily verified that the map χ from $\mathfrak{G}$ to the unit circle given by $\chi(g) = B_j(g(z))/B_j(z)$ is independent of the choice of $z \in U$; moreover this map is a character of $\mathfrak{G}$: $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$. In the terminology of $[2]$, the function B_j is *automorphic with character* χ, i.e., $B_j(g(z)) = \chi(g)B_j(z)$ for all $g \in \mathfrak{G}$ and all $z \in U$. If we now invoke Lemma 5 of $[2]$, there is a unit C_j of $H_\infty[U]$ which is automorphic with character $\bar{\chi}$: $C_j(g(z)) = \bar{\chi}(g)C_j(z)$. Then $|C_j| > \delta'$ on the set E_j'' for some $\delta' > 0$. The function b_j' defined by $b_j'(z) = C_j(z)B_j(z)$ is easily verified to be invariant under the group $\mathfrak{G}$. Thus if we define b_j on R by $b_j = b_j' \circ \Upsilon^{-1}$, b_j is a well-defined element of $H_\infty[R]$ which vanishes on the sets E_k for $k \neq j$ and whose modulus is at least $\delta\delta'$ on the set E_j. Thus, we have established the sufficiency of our condition.

The necessity follows from general Banach space considerations; one may consult $[3,$ p. $196]$ for the necessary argument.

III. From the theorem of the preceding section, we shall derive a second characterization of interpolation sets in R. We have that Γ, the boundary of R, consists of a finite collection $\{\gamma_1, \cdots, \gamma_m\}$ of pairwise disjoint analytic simple closed curves. For $j = 1, \cdots, m$, let γ_j' be a simple closed curve in R which, together with γ_j, bounds an annular region A_j in R. We suppose the A_j to have disjoint closures.

THEOREM. *The discrete set $E \subset R$ is an interpolation set for R if and only if $E_j = A_j \cap E$ is an interpolation set for A_j for each j.*

PROOF. If E is an interpolation set for R, then, *a fortiori*, E_j is one for A_j.

To establish the opposite implication, we shall show that if each E_j is an interpolation set for A_j, then for some fixed $\delta > 0$ and for all $z \in E$, $d_R(z, E) > \delta$ so that the previous result may be invoked to yield the result.

By hypothesis, R is obtained from a compact surface R_0 by deleting closed discs $V_1, \cdots, V_m$ where, say, the boundary of V_j is the curve γ_j. Let $R_j = R \cup V_1 \cup \cdots \cup V_{j-1} \cup V_{j+1} \cup \cdots \cup V_m$ so that R_j is a finite open Riemann surface which contains R. Let $\pi_1: R_1 \rightarrow U$ be an Ahlfors map for the surface R_1. Since π_1 acts as a local homeomorphism in a neighborhood of each point of γ_1, it follows that for some annulus A_1' contained in A_1 and having γ_1 as one of its bounding components, we may write $A_1' = Q_1 \cup \cdots \cup Q_M$ where for later notational convenience we take our indices to be integers mod M and where

(i) each Q_j is a closed curvilinear quadrilateral the boundary of which meets γ_1 in an arc,

(ii) Q_j and Q_{j+1} intersect in an arc connecting the components of the boundary of A_j', and

(iii) π_1 is one-to-one on $\cup \{Q_j : j = j_0, \cdots, j_0 + 5\}$ for each j_0.

Since E_1 differs from $E \cap A_1'$ by at most a finite set, E_1 is an interpolation set for R_1 if and only if $E \cap A_1'$ is. We shall show $E \cap A_1'$ to be an interpolation set for R_1.

The hypotheses on the Q_j together with the hypothesis on the set E imply, as in the previous proof, that $E \cap (Q_{i+1} \cup Q_i \cup Q_{i+1})$ is an interpolation set for the surface R_1. Consequently, there is $\beta_1 > 0$ such that for each $i = 1, \cdots, M$, and for each $z \in E \cap Q_i$, there is $f_{i,z} \in H_\infty[R_1]$ which is bounded by one, which is at least β_1 in modulus at z and which vanishes at all points of the set $E \cap (Q_{i-1} \cup Q_i \cup Q_{i+1})$ other than z.

Next, we assert the existence of functions $g_1, \cdots, g_M \in H_\infty[R_1]$ such that for some $\alpha_1 > 0$, $|g_j(z)| > \alpha_1$ if $z \in E \cap Q_j$ but $g_j(z) = 0$ if $z \in E \cap Q_k$ for $k \neq j-1, j, j+1$. Suppose the desired g_j exist. It is clearly no loss of generality to suppose them bounded by one on R_1. If $z \in E \cap A_1'$, then $z \in Q_j$ for some j. We have $|g_j(z) f_{j,z}(z)| > \alpha_1 \beta_1$, $\|g_j f_{j,z}\| < 1$, and that $g_j f_{j,z}$ vanishes on the set $(E \cap A_j') - \{z\}$. Thus the set $E \cap A_j'$ is an interpolation set for R_1.

To prove the existence of the desired functions g_j, we argue as follows. Let p and q be interior points of the arcs $Q_{j-1} \cap \partial R_1$ and $Q_{j+1} \cap \partial R_1$ respectively. Let λ be the arc of ∂R_1 containing $Q_j \cap \partial R_1$ and having p and q as end points. Let λ' be an arc in $R_0 - \overline{R_1}$ with the same end points. (Recall that R_0 is the originally given compact surface.) The arcs λ and λ' jointly constitute the boundary of a topological disc D. The surface R_1' obtained by forming the union of R_1, of D, and of the

interior of λ is again a finite open Riemann surface. It contains Q_j as a compact subset. Using an Ahlfors map for R_1', we find that the set $F_j = E \cap (Q_1 \cup \cdots \cup Q_{j-2} \cup Q_{j+2} \cup \cdots \cup Q_M)$ is a certain finite union of interpolation sets for R_1'. Thus, this set is surely contained in the zero set of some nonzero element of $H_\infty[R_1']$. The fact that Q_j is compactly contained in R_1' now implies the existence of a function in $H_\infty[R_1']$ which is zero free on Q_j and which vanishes on the set F_j. The restriction of this function to R_1 will serve as the function g_j which we seek.

Thus, $E \cap A_1'$ is an interpolation set for R_1 and so $E \cap A_1$ is also one. Similar remarks hold for the sets $E \cap A_j$ and the surfaces R_j. It follows that there exist functions $h_j \in H_\infty[R_j]$ which are bounded by one, which vanish on $E \cap A_j$ and which, for some $\delta > 0$ independent of j, satisfy $|h_j| > \delta$ on $E \cap A_k$ for $k \neq j$. By making δ sufficiently small we can also assert the existence, for each $z \in E \cap A_j$ of a function $h_{j,z}$ bounded by one on R which is at least δ in modulus at z and which vanishes at every other point of $E \cap A_j$. If $z \in E \cap A_j$, then the function $h_1 \cdots h_{j-1} h_{j,z} h_{j+1} \cdots h_m$ is bounded by one, is at least δ^m in modulus at z, and vanishes on the set $(E \cap (\cup A_j)) - \{z\}$. This estimate is uniform in z, so we may conclude from the previous theorem that $E \cap (\cup A_j)$ is an interpolation set for R. Consequently, E itself is an interpolation set for R.

IV. As noted at the beginning, these results are not new; it is the proofs which are our concern here. In the original development [5], our first theorem was obtained as an immediate corollary of our second, but the proof of the second was much more complicated than the present arguments. Several other characterizations of interpolation sets can be derived from these results as in [5].

References

1. L. V. Ahlfors, *Open Riemann surfaces and extremal problems on compact subregions*, Comment. Math. Helv. **24** (1950), 100–134.

2. F. Forelli, *Another proof of the corona conjecture for finite open Riemann surfaces*, Illinois J. Math. **10** (1966), 367–380.

3. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

4. H. L. Royden, *The boundary values of analytic and harmonic functions*, Math. Z. **78** (1962), 1–24.

5. E. L. Stout, *Bounded holomorphic functions on finite Riemann surfaces*, Trans. Amer. Math. Soc. **120** (1965), 255–285.

6. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.

YALE UNIVERSITY