THE ESSENTIAL SET OF FUNCTION ALGEBRAS
ROBERT E. MULLINS

Let $X$ be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex-valued continuous functions on $X$ under the sup-norm. By a function algebra $A$ we mean a closed subalgebra of $C(X)$ which separates points and contains the constant functions. Let $S(A)$ denote the space of maximal ideals of $A$ and let $E(A)$ denote the essential set of $A$ as defined by Bear [1], i.e. $E(A)$ is the hull of the largest closed ideal of $C(X)$ which is contained in $A$. We will obtain another characterization of the essential set in the case when $S(A)=X$ and thereby obtain some results of a global nature from local hypotheses.

**Theorem 1.** Let $A$ be a function algebra on a compact metric space $X$. Suppose $X=S(A)$ and let $E$ be the essential set of $A$ in $X$. If $x_0$ has a neighborhood $V$ such that $A|V=C(V)$ then $x_0\in E$.

**Proof.** Let $V$ be a neighborhood of $x$ such that $A|V=C(V)$. Then $x_0$ is a local peak point, and hence [2, Theorem 4.1] $x_0$ is a global peak point. Let $f\in A$, $f(x_0)=1$ and $|f(x)|<1$ if $x\neq x_0$. Let $r$ be a number so close to one that

$$U = \{x: \text{Re } f(x) \geq r\} \subset V.$$ 

Then $\text{Re } f(x) \leq s < 1$ for all $x\in X\sim V$. Let $p_n$ be a sequence of polynomials which converge uniformly on $f[U]\cup f[X\sim V]$ to a function one on $f[U]$ and zero on $f[X\sim V]$ (see e.g. Wermer: Banach algebras and analytic functions, Theorem 7.6). Then $p_n\circ f$ converges uniformly on $X$, except possibly on $V\sim U$, to a function one at $x_0$ and zero off $V$. Let $g\in A$, $g(x_0)=1$ and $g=0$ on $V\sim U$. Then $g(p_n\circ f)$ converges uniformly on $X$ to a function $k\in A$ such that $k(x_0)=1$, and $k=0$ off $U$. Let $W$ be a neighborhood of $x$ such that $k\neq 0$ on $W$. Let $h$ be any continuous function on $X$ which is zero off $W$. Let $w\in A$ with $w=h/k$ on $W$. Then $h=wk\in A$. That is, $A$ contains every continuous function zero off $W$. Hence $X\sim W\supset E$, and $x_0\in E$.

**Corollary.** Let $A$ be a function algebra on the compact metric space $X=S(A)$ and let $E$ be the essential set of $A$ in $X$. Then $E=X\sim P$, where $P=\{x\in X: A|V=C(V)\text{ for some neighborhood }V\text{ of }x\}$.

Received by the editors January 27, 1966.

1 Part of the material in this paper appears in the author's doctoral dissertation at Northwestern University. This research was supported in part by the National Science Foundation.

271
Theorem 2. Let \( A \) be a function algebra on a compact metric space \( X = S(A) \). Let \( F_1, F_2, \cdots \) be a sequence of closed sets such that \( X = \bigcup_{i=1}^{\infty} F_i \) and \( A \mid F_i \) is closed in \( C(F_i) \) for \( i = 1, 2, \cdots \). Then \( \bigcup_{i=1}^{\infty} E_i \) is the essential set of \( A \) in \( X \) where \( E_i \) is the essential set of \( A \mid F_i \) in \( F_i \).

Proof. Let \( E \) denote the essential set of \( A \) in \( X \) and let \( I \) denote the largest closed ideal of \( C(X) \) contained in \( A \). Let \( J_i \) be the largest ideal of \( C(F_i) \) which is contained in \( A \mid F_i \). Clearly \( I \mid F_i \) is contained in \( J_i \). Thus for \( i = 1, 2, \cdots \), \( E_i = \{ x \in F_i : f(x) = 0 \text{ for all } f \in J_i \} \) is contained in \( E = \{ x \in X : f(x) = 0 \text{ for all } f \in I \} \). Since \( E \) is closed, \( \bigcup_{i=1}^{\infty} E_i \subseteq E \). Let \( U = E \sim \bigcup_{i=1}^{\infty} E_i \). It is shown in Bear [1] that \( A \mid E \) is a function algebra with essential set \( E \) and maximal ideal space \( E \). Since \( C = \bigcup_{i=1}^{\infty} (U \cap F_i) \) is open in \( E \), it follows from the Baire Category theorem that some \( U \cap F_i \) has a nonempty interior in \( E \). Let \( V \) be a closed neighborhood, relative to \( E \), which is contained in \( U \cap F_i \). Since \( V \) is disjoint from \( E_i \), the essential set of \( A \mid F_i \), it follows that \( A \mid V = C(V) \). This contradicts the result of Theorem 1, namely that \( P = \{ x \in E : A \mid V_x = C(V_x) \text{ for some closed neighborhood } V_x \text{ of } x \} \) must be the empty set.

Corollary. Let \( A \) be a function algebra on the compact metric space \( X = S(A) \). Let \( F_1, F_2, \cdots \) be a sequence of closed sets such that \( X = \bigcup_{i=1}^{\infty} F_i \) and \( A \mid F_i = C(F_i) \) for \( i = 1, 2, \cdots \). Then \( A = C(X) \).

If \( X \) is a union of a finite number of \( F_i \) and if \( A \mid F_i = C(F_i) \), it is a necessary consequence that \( S(A) = X \) as the following lemma shows.

Lemma. Let \( X \) be a compact Hausdorff space and let \( F_1, \cdots, F_n \) be \( n \) closed sets such that \( X = \bigcup_{i=1}^{\infty} F_i \). If \( A \) is a function algebra on \( X \) such that \( A \mid F_i \) is closed in \( C(F_i) \) and \( S(A \mid F_i) = F_i \) for \( i = 1, 2, \cdots, n \) then \( S(A) = X \).

Proof. Let \( M \) be a proper maximal ideal in \( A \). If \( M \mid F_i = A \mid F_i \) for \( i = 1, 2, \cdots, n \), there exist functions \( f_1, \cdots, f_n \) in \( M \) such that \( f_i(x) = 1 \) whenever \( x \) is in \( F_i \). Let \( h = f_1 + f_2 - f_1 f_2 \). Assuming \( h \) has been defined let \( h_{j+1} = f_{j+1} + h_j - f_j h_j \). We thus get a function \( h_n \) in \( M \) such that \( h_n \) is one on \( X \). This contradicts the assertion that \( M \) is a proper ideal. There must therefore exist an integer \( k \) (\( k \leq n \)) such that \( M \mid F_k \) is a proper ideal in \( A \mid F_k \). The ideal \( M \mid F_k \) is contained in a maximal ideal of \( A \mid F_k \). Thus there exists \( x_0 \) in \( F_k \) such that \( M \mid F_k \subseteq \{ f \in A : f(x_0) = 0 \} \). It then follows that \( M \) corresponds to evaluation at \( x_0 \). This completes the proof that \( S(A) = X \).

Theorem 3. Let \( A \) be a function algebra on a compact metric space \( X \).
Let $F_1, \ldots, F_n$ be $n$ closed sets such that $X = \bigcup_{i=1}^{n} F_i$ and $A \mid F_i = C(F_i)$ for $i = 1, \ldots, n$. Then $A = C(X)$.

**Proof.** Since $S(A) = X$ by the above lemma, this theorem is precisely the corollary after Theorem 2 in the finite case.

**Corollary.** Let $A$ be a function algebra on a compact metric space $X$. If for each $x$ in $X$ there exists a closed neighborhood $V_x$ of $x$ such that $A \mid V_x = C(V_x)$, then $A = C(X)$.

I wish to thank the referee for supplying a proof of Theorem 1, much shorter than the proof I originally submitted.

**References**


Northwestern University and
Marquette University