FINITE \( p \)-SOLVABLE LINEAR GROUPS WITH
A CYCLIC SYLOW \( p \)-SUBGROUP

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In [3] N. Itô proved that if \( p \) is a prime and if \( G \) is a finite \( p \)-solvable linear group over the complex number field of degree less than \( p - 1 \), then \( G \) has a normal abelian Sylow \( p \)-subgroup. In this paper the same type of proof (see also [1, p. 143]) will be used for the following result.

**Theorem.** Let \( p \) be a prime and let \( G \) be a finite \( p \)-solvable group which contains a cyclic Sylow \( p \)-subgroup of order \( p^a \). If \( G \) has a faithful representation over the complex number field of degree less than \( p^{a-t}(p-1) \) where \( t \) is an integer such that \( 1 \leq t \leq a \), then \( G \) has a normal subgroup of order \( p^t \).

Let \( G \) be a counterexample to the theorem of minimal order. Let \( P \) be a fixed Sylow \( p \)-subgroup of \( G \) of order \( p^a \), \( P_0 \) the unique subgroup of \( P \) of order \( p^t \). Clearly, \( G \) is not a \( p \)-group. Let \( |G| \) have at least three distinct prime divisors. Since \( G \) is \( p \)-solvable, \( G \) contains a \((p, q)\)-Hall subgroup \( S(p, q) \) for any prime \( q \) which is distinct from \( p \) [4, page 196]. Since \( G \neq S(p, q) \), \( P_0 \triangleleft S(p, q) \) by the induction hypothesis. Since this is true for any \( q \) which is distinct from \( p \), \( P_0 \triangleleft G \). Hence we may assume that \( |G| = p^{a-b} \) for some prime \( q \neq p \).

From now on \( \chi \) will denote a fixed faithful character of \( G \) (i.e., a character of a faithful representation of \( G \)) of minimal degree. A contradiction will be obtained after a series of short steps.

(1) \( \chi \) is irreducible.

**Proof.** Suppose \( \chi \) is reducible. Let \( \chi_1 \) be a nonlinear irreducible constituent of \( \chi \) and let \( K \) be its kernel. Then \( 1 \neq K \neq G \) and since \( \chi_1 \) is nonlinear, \( PK \neq G \).

It can now be seen that \( P_0 K \triangleleft G \). If \( P_0 \leq K \), this is clear. Hence assume that \( |K| = p^c q^d \), \( 0 \leq c < t \), \( 0 \leq d \leq b \). Since \( \chi_1 \) is a faithful character of \( G/K \) of degree less than \( p^{a-t}(p-1) = p^{(a-c)-(t-c)}(p-1) \), the induction hypothesis implies that \( G/K \) contains a normal subgroup of order \( p^{t-c} \). Let \( P_1 \leq P \) be such that \( P_1 K/K \) is this group. Then \( P_1 K \triangleleft G \) and \( |P_1| = p^t \) and hence \( P_1 = P_0 \).

The induction hypothesis implies that \( P_0 \triangleleft PK \). Therefore \( P_0 \triangleleft P_0 K \) whence \( P_0 \triangleleft G \). This contradiction proves (1).

Let \( Q \) be a Sylow \( q \)-subgroup of \( G \).

(2) \( Q \triangleleft G \).

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Proof. Let $Q_0$ be the maximal normal $q$-subgroup of $G$. Then $G/Q_0$ contains no normal $q$-subgroup. Let $P_1 \leq P$ be such that $P_1 Q_0 / Q_0$ is the maximal normal $p$-subgroup of $G/Q_0$. By Lemma 1.2.3 of [2], $P_1 Q_0 / Q_0$ contains its centralizer in $G/Q_0$. Hence, $P \leq P_1 Q_0$ and so $P = P_1$ and $P Q_0 < G$. The induction hypothesis forces $P Q_0 = G$. Hence, $Q_0$ is a Sylow $q$-subgroup of $G$ as was to be shown.

(3) $G$ contains no normal $p$-subgroup.

Suppose on the contrary that $G$ contains a proper normal $p$-subgroup $U$. Then $P \leq C(U) < G$. If $C(U) = G$, then $P_0 < C(U)$ and so $P_0 < G$, a contradiction. Therefore $C(U) = G$ and $p \mid |Z(G)|$ where $Z(G)$ is the center of $G$. Let $P_1 = P \cap Z(G)$. Then $\chi_{|P_1} = \chi(1) \mu$ where $\mu$ is a linear character of $P_1$. Let $\lambda$ be a linear character of $G/Q$ such that $\lambda_{|P_1} = \mu$. Then $\lambda \chi$ is a faithful character of $G/P_1$. The induction hypothesis yields that (since $P_1 \neq P_0$) $P_0 / P_1 < G / P_1$ and hence $P_0 < G$, a contradiction.

(4) $\chi_{|Q}$ is irreducible. Hence $Q$ is nonabelian.

Suppose $\chi_{|Q}$ is reducible. Let $P_1$ be the maximal subgroup of $P$ such that $\chi_{|P_1 Q}$ is reducible. Since $\chi$ is an irreducible character of $G$, $P_1 \neq P$. Let $P_2$ be the unique subgroup of $P$ such that $|P_2 : P_1| = p$. By maximality of $P_1$, $\chi_{|P_2 Q}$ is irreducible. By [1, pp. 54–55], $\chi_{|P_2 Q}$ is a sum of $p$ distinct conjugate characters and if $\theta$ is one of these, the inertia group of $\theta$ in $P_2 Q$ is $P_1 Q$. But this implies that the inertia group of $\theta$ in $G$ is $P_1 Q$. Therefore the induced character $\theta^*$ is irreducible and $\chi = \theta^*$. Hence $\chi(1) = |G : P_1 Q| \theta(1) = p \theta(1)$ and so $|G : P_1 Q| = p$. Now $\theta(1) = \chi(1) / p < p^{(a-1) - t(p-1)}$. Thus $\chi_{|P_1 Q}$ is a faithful character of $P_1 Q$ all of whose irreducible constituents have degree less than $p^{(a-1)-t(p-1)}$. Let $K_1, \ldots, K_p$ be the kernels of these constituents. If $t \leq a - 1$, the induction hypothesis implies that $K_i P_0 < P_1 Q$ for all $i$. If this is the case, then $P_0 = K_1 P_0 \cap K_2 P_0 \cap \cdots \cap K_p P_0 < P_1 Q$ and so $P_0 < G$. Since this cannot occur, $t > a - 1$ or $t = a$, contradicting [3]. This proves (4).

(5) If $Q_0$ is a proper subgroup of $Q$ normal in $G$, then $Q_0 \leq Z(G)$.

Proof. Since $Q_0 < G$, $P Q_0$ is a group and $P Q_0 \neq G$ since $Q_0 \neq Q$. By induction $P_0 < P Q_0$. This implies that $P_0 Q_0 = P_0 \times Q_0$ and $P_0 \leq C(Q_0)$ < $G$. Suppose $C(Q_0) = G$. Then $\chi_{|C(Q_0)}$ must be reducible for otherwise Schur's Lemma would imply $Q_0 \leq Z(G)$, each element of $Q_0$ commuting with the irreducible system $C(Q_0)$. $P$ is not contained in $C(Q_0)$ since otherwise induction yields $P_0 < C(Q_0)$ and so $P_0 < G$.

Suppose $P C(Q_0) = G$. Then $P_0 < P C(Q_0)$ and hence $P_0 < C(Q_0)$ which is not the case. Therefore, $P C(Q_0) = G$ and $|G : C(Q_0)|$ is a power of $p$. This implies that $\chi_{|Q}$ is reducible, contradicting (4). (5) is now proved.
As $G$ has no normal subgroup of index $q$, neither does $G/Q'$. Now $G/Q'$ has an abelian $q$-Sylow subgroup. Suppose there exists $Q_0 < G$ with $Q' < Q_0 < Q$. By (5) $Q_0 \leq Z(G)$ and $Q_0/Q'$ is a subgroup of the center of $G/Q'$. Therefore [5, page 173] $G/Q'$ contains a normal subgroup of index $q$, which is a contradiction. This proves

(6) $Q/Q'$ is a minimal normal subgroup of $G/Q'$.

(7) Let $N(P)$ and $C(P)$ be, respectively, the normalizer and the centralizer of $P$ in $G$. Then $N(P) = C(P) = P \times Q'$.

Proof. Let $Q_1 = N(P) \cap Q$. Then $N(P) = P \times Q_1 \leq C(P)$. Hence, $N(P) = C(P)$. $Q' \leq Q_1$ by (5) and so $Q_1 < G$ since $Q/Q'$ is abelian. Therefore, $Q_1 < G$. By (6), $Q_1 = Q'$, proving (7).

By (5), $Q' \leq Z(Q)$ and by (6), $Q' = Z(Q)$. Since $Q' \leq Z(G)$, $Q'$ is cyclic, $\chi$ being irreducible and faithful. By (6), $Q/Q'$ has exponent $q$. Hence by [1, p. 142],

(8) $|Z(Q)| = q$, $|Q: Q'| = q^{2n}$ for some integer $n$ and every nonlinear character of $Q$ has degree $q^n$.

Since the cyclic group $P$ is faithful on the chief factor $Q/Q'$, it follows that $p^a$ divides $q^{2n} - 1$, and so $q^n \equiv \pm 1 \mod p^a$. This implies that

\[ p^a \sim q^n + 1 = \chi(1) + 1 < p^{a-t}(p - 1) + 1 = p^{a-t+1} - p^{a-t} + 1 < p^{a-t+1} \sim p^a, \]

a final contradiction.

References