ANOTHER S-ADMISSIBLE CLASS OF SPACES
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For a topological space $X$, $S(X)$ denotes the semigroup of all continuous functions mapping $X$ into itself where the binary operation is that of composition. A class of topological spaces was defined in [4] to be $S$-admissible if for any two spaces $X$ and $Y$ of the class, every isomorphism from $S(X)$ onto $S(Y)$ is induced by a homeomorphism. By this, we mean that for each isomorphism $\phi$ from $S(X)$ onto $S(Y)$, there exists a homeomorphism $h$ from $X$ onto $Y$ such that $\phi(f) = h \circ f \circ h^{-}$ for each $f$ in $S(X)$. Hence, the $S$-spaces defined in [2] form an $S$-admissible class of spaces. We refer to a topological space $X$ as an $S^*$-space if it is $T_1$ and for each closed subset $F$ of $X$ and each point $p$ in $X - F$ there exists a function $f$ in $S(X)$ and a point $y$ in $X$ such that $f(x) = y$ for each $x$ in $F$ and $f(p) \neq y$. Our main purpose here is to prove the following

**Theorem 1.** The class of $S^*$-spaces is an $S$-admissible class.

**Proof.** Let $X$ and $Y$ be $S^*$-spaces and let $\phi$ be an isomorphism from $S(X)$ onto $S(Y)$. It follows from Theorem 2.4 of [3] that there exists a bijection $h$ from $X$ onto $Y$ such that $\phi(f) = h \circ f \circ h^{-}$ for each $f$ in $S(X)$. Since the existence of the bijection $h$ is not hard to prove, we do so here for the sake of completeness. For a point $x$ in $X$, we use the symbol $x$ to denote the constant function defined by $x(y) = x$ for each $y$ in $X$. Since a function $g$ in $S(X)$ has the property $g \circ f = g$ for each $f$ in $S(X)$ if and only if $g$ is a constant function, it follows that the set of all constant functions of $S(X)$ is precisely the ideal of left zeros of $S(X)$. Then, for any point $x$ in $X$, $x$ is a left zero of $S(X)$ and thus $\phi(x)$ is a left zero of $S(Y)$, i.e., $\phi(x) = y$ for some $y$ in $Y$. Define $h(x) = y$. Since $\phi$ maps the ideal of left zeros of $S(X)$ bijectively onto the ideal of left zeros of $S(Y)$, $h$ is a bijection from $X$ onto $Y$. Using the fact that for any $x$ in $X$, $\phi(x) = h(x)$, we see that for $f$ in $S(X)$ and $y$ in $Y$,

\[
(h \circ f \circ h^{-})(y) = h(f(h^{-}(y))) = h(f(h^{-}(y)))(y) \\
= \phi(f(h^{-}(y)))(y) = \phi(f \circ h^{-}(y))(y) \\
= (\phi(f) \circ \phi(h^{-}(y)))(y) = (\phi(f) \circ y)(y) = \phi(f)(y).
\]

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Thus, $\phi(f) = h \circ f \circ h^{-1}$.

Now for any point $z$ in $X$ and any function $f$ in $S(X)$, we let

$$H(z, f) = \{x \in X : f(x) = z\},$$

and we show that

$$h[H(z, f)] = H(h(z), \phi(f)).$$

This is a consequence of the fact that the following statements are successively equivalent. $y \in h[H(z, f)], y = h(x)$ and $f(x) = z, \phi(x) = y$ and $\phi(f) \circ \phi(x) = \phi(f \circ x) = \phi(f(x)) = \phi(z), \phi(f) \circ y = h(z), \phi(f)(y) = h(z), y \in H(h(z), \phi(f)).$

The proof will therefore be complete when we show that for an $S^*$-space $X$, the family of sets of the form $H(z, f)$ is a basis for the closed subsets of $X$. Since $X$ is $T_1$, any set of the form $H(z, f)$ is closed. Now let $F$ be a proper closed subset of $X$ and let $x$ be a point of $X - F$. Then there exists a function $f_x$ in $S(X)$ and a point $y_x$ in $X$ such that $f_x(z) = y_x$ for each $z$ in $F$ and $f_x(x) \neq y_x$. It follows that

$$F = \bigcap \{H(y_x, f_x) : x \in X - F\}$$

and the theorem is proved.

The following two results indicate that the class of $S^*$-spaces is reasonably extensive.

**Theorem 2.** Every 0-dimensional Hausdorff space is an $S^*$-space.

**Proof.** Suppose $F$ is a closed subset of $X$ and $p$ is a point in $X - F$. Then there exists a set $G$ which is both open and closed such that $p \in G \subset X - F$. Choose any point $q$ other than $p$ and define $f(x) = p$ for $x$ in $G$ and $f(x) = q$ for $x$ in $X - G$. Then $f$ is continuous, which proves $X$ is an $S^*$-space.

**Theorem 3.** Every completely regular Hausdorff space containing at least two distinct points which are connected by an arc is an $S^*$-space.

**Proof.** Suppose $F$ is a closed subset of $X$ and $p \in X - F$. Since $X$ is completely regular, there exists a continuous function $f$ mapping $X$ into the closed interval $I$ such that $f(x) = 0$ for $x$ in $F$ and $f(p) = 1$. Let $q_1$ and $q_2$ be two distinct points of $X$ which are connected by an arc. Then there exists a continuous function $g$ from $I$ into $X$ such that $g(0) = q_1$ and $g(1) = q_2$. Therefore, $g \circ f$ is a function in $S(X)$ with the properties $g \circ f(x) = q_1$ for $x$ in $F$ and $g \circ f(p) \neq q_1$.

Let $X$ be any completely regular, Hausdorff space and let $Y = X \cup I$. We assume $X$ and $I$ are disjoint (suitable modifications can be made
otherwise). Take as a basis for the open sets of $Y$ those sets which are open either in $X$ or in $I$. Then by the previous theorem, $Y$ is an $S^*$-space and we have

**Corollary 4.** Every completely regular Hausdorff space is a subspace of an $S^*$-space.

There exist completely regular Hausdorff spaces which are not $S^*$-spaces. In fact, Cook [1] has given an example of a compact, metric, one-dimensional, indecomposable continuum $K$ such that $S(K)$ consists precisely of the constant functions and the identity function. $K$ is certainly not an $S^*$-space. Moreover, no $S$-admissible class can contain $K$ since any bijection from $S(K)$ onto $S(K)$ which maps the identity onto itself is an isomorphism and only one of these is induced by a homeomorphism. By the previous corollary, $K$ is a subspace of an $S^*$-space. Thus, the property of being an $S^*$-space is not hereditary.

An open set $G$ containing a point $x$ in $X$ was defined in [2] to be an $S$-neighborhood if

(i) $G$ consists of $X$ alone or

(ii) there exists a continuous function $f$ from $\text{cl} \ G$ into $X$ such that $f(x) \neq x$ and $f(y) = y$ for each $y$ in $\text{cl} \ G - G$.

A space $X$ was then defined to be an $S$-space if each point has a basis of $S$-neighborhoods. It was shown in [2] that the class of $S$-spaces is $S$-admissible. Once more, let $K$ be the space described by Cook, let $I$ be the closed unit interval and topologize $Y = K \cup I$ by taking as a basis for open sets those sets which are open either in $X$ or in $I$. Choose $x$ in $K$ and let $G$ be any open subset of $K$ containing $x$ such that $K - G$ consists of more than one point. Now suppose there exists a continuous function $f$ mapping $\text{cl} \ _Y \ G$ into $Y$ such that $f(x) \neq x$ and $f(y) = y$ for each $y$ in $\text{cl} \ _Y \ G - G$. Define a function $g$ from $Y$ into $Y$ by $g(y) = y$ if $y \in Y - G$ and $g(y) = f(y)$ for $y \in G$. Then $g$ is continuous and since $K$ is connected, $g[K] \subset K$. This, together with the facts that $g[K] \cap K \neq \emptyset$, and $K$ and $I$ are both open subsets of $Y$, implies $g[K] \subset K$. This, however, is a contradiction since $g$ is neither the identity function nor a constant function. This implies that the point $x$ has no $S$-basis and therefore that $Y$ is not an $S$-space. It follows from Theorem 3, however, that $Y$ is an $S^*$-space. Therefore, there exist $S^*$-spaces which are not $S$-spaces. We do not know if every $S$-space must be an $S^*$-space.

We conclude by mentioning that the space described in Example (2.7) of [2] is another example of a space which is not an $S^*$-space.
REFERENCES


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