BOUNDS FOR SOLUTIONS OF 2ND ORDER COMPLEX DIFFERENTIAL EQUATIONS

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1. It is well known [1], [2] that upper and lower bounds for the norm of a solution of ordinary differential systems can be given in terms of solutions of related first order scalar equations. However, the independent variable \( t \) is taken to be real there. In [3] upper bounds for solutions of a class of 2nd order complex differential equations were obtained.

In this paper we derive upper as well as lower bounds for solutions of the complex differential equation

\[
y'' + y + yf(y, y', z) = 0,
\]

where \( f \) is an entire function of \( y \) and \( y' \), analytic in \( z \) for \( |z| < R \).

Let \( Y \) denote the column vector with components \( y, y' \) and let \( \bar{f} \) denote the function of \( Y \) and \( z \) which takes the values \( f(y, y', z) \), that is,

\[
\bar{f}(Y, z) = f(y, y', z).
\]

(1) is equivalent to

\[
Y' = AY + B(Y, z)Y,
\]

where \( A, B(Y, z) \) are the matrices

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
-\bar{f} & 0
\end{pmatrix}
\]

respectively.

2. We use the absolute value norm; namely, for a vector \( Y \) with components \( y, y' \),

\[
|Y| = |y| + |y'|.
\]

**Lemma 1.** Suppose that there is a continuous, nonnegative function \( g(s, t) \) defined on the half-strip \( \{(s, t) \mid 0 \leq s < \infty, 0 \leq t < R\} \), such that

\[
|\bar{f}(Y, z)| \leq g(|Y|, |z|).
\]

Let \( y(z) \) be a solution of (1) for which

\[
|y(0)| = a, \quad |y'(0)| = b, \quad a + b > 0.
\]

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and let $s(t)$ be the maximal solution of

$$ds/dt = s(1 + g(s, t)),$$

satisfying $s(0) = a + b$. Then, for all $t < R$ such that $s(t)$ exists, $R$ being assumed sufficiently large, we have

$$|Y(z)| \leq s(t), \quad t = |z|.$$

**Proof.** Let $\Phi(z)$ be the fundamental matrix of $Y' = AY$; that is,

$$\Phi(z) = \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix}.$$

A solution of (2) which satisfies $Y(0) = Y_0$ is

$$Y(z) = \Phi(z) Y_0 + \int_0^z \Phi(z) \Phi^{-1}(\xi) B(Y(\xi), \xi) Y(\xi) d\xi,$$

where the integration is carried out along the ray $\theta = \theta_0$. Let us write

$$z = t \exp(i\theta_0), \quad \xi = \tau \exp(i\theta_0).$$

Then (5) can be written as

$$Y(t, \theta_0) = \Phi(t, \theta_0) Y_0 + \exp(i\theta_0)$$

$$\cdot \int_0^t \Phi(t, \theta_0) \Phi^{-1}(\tau, \theta_0) B(Y(\tau, \theta_0), \tau \exp(i\theta_0)) Y(\tau, \theta_0) d\tau,$$

where $Y(\cdot, \exp(i\theta_0)) \equiv Y(\cdot, \theta_0)$. Also, if $h > 0$,

$$Y(t + h, \theta_0) = \Phi(t + h, \theta_0) Y_0 + \exp(i\theta_0)$$

$$\cdot \int_0^{t + h} \Phi(t + h, \theta_0) \Phi^{-1}(\tau, \theta_0) B(Y(\tau, \theta_0), \tau \exp(i\theta_0)) Y(\tau, \theta_0) d\tau.$$

If we let $m(t, \theta_0) = |Y(t, \theta_0)|$, then

$$m(t + h, \theta_0) - m(t, \theta_0)$$

$$\leq |Y(t + h, \theta_0) - Y(t, \theta_0)|$$

$$\leq |Y(t + h, \theta_0) - \Phi(t + h, \theta_0) \Phi^{-1}(t, \theta_0) Y(t, \theta_0)|$$

$$+ |\Phi(t + h, \theta_0)(\Phi^{-1}(t + h, \theta_0) - \Phi^{-1}(t, \theta_0)) Y(t, \theta_0)|$$

$$= |\Phi(t + h, \theta_0) \int_t^{t + h} \Phi^{-1}(\tau, \theta_0) B(Y(\tau, \theta_0), \tau \exp(i\theta_0)) Y(\tau, \theta_0) d\tau|$$

$$+ |\Phi(t + h, \theta_0)(\Phi^{-1}(t + h, \theta_0) - \Phi^{-1}(t, \theta_0)) Y(t, \theta_0)|.$$
\[ \Phi(t, \theta_0) \frac{d\Phi^{-1}}{dt} (t, \theta_0) Y(t, \theta_0) \]

\[ = -\exp(i\theta_0) A y(t, \theta_0) = \exp(i\theta_0) \begin{pmatrix} -y'(t \exp(i\theta_0)) \\ y(t \exp(i\theta_0)) \end{pmatrix}, \]

we get

\[ \dot{m}_+(t, \theta_0) \leq m(t, \theta_0)(1 + g(m(t, \theta_0), t)), \]

where \( \dot{m}_+ \) is the right-hand derivative of \( m \).

Hence the conclusion follows from Theorem 4.1, p. 26 of [4], in view of the arbitrariness of \( \theta_0 \).

**Theorem 2.** Let the hypotheses of Lemma 1 be satisfied. Then,

\[ |y(z)| \leq e^{-t} \left[ a + \int_0^t s(\tau)e^{\tau}d\tau \right], \quad t = |z|. \]

**Proof.** By Lemma 1,

\[ |y(\tau \exp(i\theta_0))| + |y'(\tau \exp(i\theta_0))| = |Y(\tau, \theta_0)| \leq s(\tau), \]

and so

\[ |y(\tau \exp(i\theta_0))| + \frac{d}{d\tau} \left( |y(\tau \exp(i\theta_0))| \right) \leq s(\tau). \]

Therefore,

\[ \frac{d}{d\tau} \left( e^{\tau} |y(\tau \exp(i\theta_0))| \right) \leq e^{\tau} s(\tau), \]

whence,

\[ e^t |y(t \exp(i\theta_0))| \leq a + \int_0^t s(\tau)e^{\tau}d\tau. \]

The conclusion follows since \( \theta_0 \) is arbitrary.

**Example.** If \( g(s, t) \) is of the form \( ks^n \), (4) becomes

\[ \frac{ds}{dt} = s + ks^{n+1}, \]

which can be solved explicitly. Indeed, the solution satisfying \( s(0) = a+b \) is

\[ e^t[((a + b)^{-n} + k) - ke^{nt}]^{-1/n}. \]

Thus, in particular, when \( k=n=a+b=1 \), we have for \( t<\ln 2 \),
\[
|y(z)| \leq (a + 1)e^{-t} - 1 - 2e^{-t} \ln(2 - e^t).
\]

3. In addition to the estimate (6), we get
\[
-m(t, \theta_0)(1 + g(m(t, \theta_0), t)) \leq \dot{m}_+(t, \theta_0).
\]
This leads to

**Lemma 3.** Let \(y(z)\) be a solution of (1) as in Lemma 1. Let \(\sigma(t)\) be the minimal solution of
\[
d\sigma/dt = -\sigma(1 + g(\sigma, t)),
\]
satisfying \(\sigma(0) = a + b\). Then, for all \(t < R\) such that \(\sigma(t) \geq 0\), we have
\[
|V(z)| \geq \sigma(t), \quad t = |z|.
\]

**Proof.** It is sufficient to show that, for arbitrary \(\theta_0\),
\[
m(t, \theta_0) \geq \sigma_+(t),
\]
where \(\sigma_+(t)\) is a solution of
\[
d\sigma/dt = -\sigma(1 + g(\sigma, t)) - \epsilon, \quad \epsilon > 0,
\]
satisfying the same initial condition as \(\sigma(t)\).

Suppose for some \(\epsilon > 0\), (10) is false. Then there exists \(\ell (\geq 0)\) such that
\[
m(\ell, \theta_0) = \sigma_+(\ell), \quad m(t, \theta_0) < \sigma_+(t) \quad \text{for } t > \ell;
\]
whence,
\[
\dot{m}_+(t, \theta_0) \leq (d\sigma_+/dt)(\ell) = -m(\ell, \theta_0)(1 - g(m(\ell, \theta_0), \ell)) - \epsilon,
\]
a contradiction in view of (7). This completes the proof.

Before we turn to the main theorem of this section, we state, as a separate lemma, the following result which we require.

**Lemma 4.** Let \(|y(z)| = \rho(t, \theta), z = te^{\theta}, and M(t) = \max_{0 \leq \theta \leq 2\pi} \rho(t, \theta)\). Then,
\[
(\partial \rho/\partial t)(t, \theta_0) \leq \dot{M}_+(t),
\]
where \(M(t) = \rho(t, \theta_0)\).

**Proof.** Let \(h > 0\) and let \(M(t + h) = \rho(t + h, \theta_0)\). Then,
\[
\frac{\rho(t + h, \theta_0) - \rho(t, \theta_0)}{h} \leq \frac{M(t + h) - M(t)}{h}.
\]
The conclusion is immediate in view of the fact that \(\partial \rho/\partial t\) exists.
Theorem 5. Let the hypotheses of Lemma 3 be satisfied. Then,

\[ M(t) \geq e^{-t} \left[ a + \int_0^t \sigma(\tau) e^{\tau} d\tau \right], \quad t = |z|. \]

Proof. Set

\[ y(z) = \rho e^{i\phi}, \quad z = t e^{i\theta}. \]

For each \( t \),

\[ i\rho e^{i\phi} y'(z) = \frac{\partial}{\partial \theta} (\rho e^{i\phi}) = e^{i\phi} \left[ \frac{\partial \rho}{\partial \theta} + i\rho \frac{\partial \phi}{\partial \theta} \right]. \]

If, for fixed \( t \), the maximum \( M(t) \) of \( \rho(t, \theta) \) is taken when \( \theta = \theta_0 \), we have

\[ (\partial \rho / \partial \theta)(t, \theta_0) = 0. \]

Therefore, (12) yields

\[ t \exp(i\theta_0) y'(z_0) = \rho e^{i\phi} (\partial \rho / \partial \theta)(t, \theta_0), \quad z_0 = t \exp(i\theta_0), \]

that is, \( \exp(i(\theta_0 - \phi))y'(z_0) = (\partial \rho / \partial \theta)(t, \theta_0) \), by the Cauchy-Riemann equations. Also, since \( \partial \rho / \partial t \) exists, it is easy to see that

\[ (\partial \rho / \partial t)(t, \theta_0) \geq 0; \]

and so,

\[ |y'(z_0)| = (\partial \rho / \partial t)(t, \theta_0). \]

Thus, from (9) and Lemma 4,

\[ M(t) + \dot{M}(t) \geq \sigma(t), \]

that is,

\[ \dot{K}(t) \geq e^t \sigma(t), \quad K(t) = e^t M(t). \]

The proof is completed by noting that (11) is the integral form of (13) since \( K(0) = a \) and \( t \geq 0 \).

Example. As earlier, if \( g(s, t) \) is \( k s^{n+1} \), (8) is

\[ d\sigma / dt = -\sigma - k \sigma^{n+1}. \]

The solution of this equation for which \( \sigma(0) = a + b \) is

\[ e^{-t} [((a + b)^{-n} + k) - ke^{-nt}]^{-1/n}. \]

Thus, when \( k = n = a + b = 1 \), we have
\[ M(t) \geq e^{-t}[a + \frac{1}{2} \ln(2e^t - 1)]. \]

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References


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