# TWO THEOREMS OF KRASNOSEL'SKIĬ TYPE[1]

F. A. VALENTINE

In 1946 Krasnosel'skiĭ proved that if every three points in the boundary of a compact set in the Euclidean plane $E_2$ can see at least one point of $S$ via $S$ then $S$ is starshaped. In his paper [2] Krasnosel'skiĭ proved the corresponding theorem for $E_n$ under still weaker hypotheses. These theorems have been recently expanded by Robkin in his doctoral dissertation [4]. In our first section we state conditions in $E_2$ sufficient for a subset $B$ of $S$ to see a point of $S$ via $S$. In the second section we characterize those bounded closed connected plane sets $S$ which can be covered by a family of parallel lines each of which intersects $S$ in a connected set.

1. Here we investigate the following type of relative visibility.

DEFINITION 1. A set $B$ is said to see a point of $S$ in $E_n$ via $S$ if there exists a point $p\in S$ such that for every point $b\in B$ it is true that $pb\subset S$, where $pb$ is the segment joining $p$ and $b$.

In order to describe such sets $B$ in the plane $E_2$, the following concept proves to be useful.

DEFINITION 2. A set $S\subset E_2$ has the four point simply-connected property if the following holds. If $x\in S$, $p\in S$, $y\in S$, $q\in S$ and if the four segments $xp$, $py$, $yq$, $xq$ all belong to $S$, then the closed bounded portion of $E_2$ which has these four edges as its boundary belongs to $S$.

It should be noted that the quadrilateral with consecutive vertices $x$, $p$, $y$, $q$ may bound either a four-sided figure or two triangular regions, the latter occurring when either $xq$ and $yp$ or $xp$ and $yq$ intersect internally. There are also degenerate cases.

*Notation.* We let cl $A$, int $A$, bd $A$, conv $A$ denote the closure, interior, boundary and convex hull of $A$ respectively. As stated earlier $xy$ denotes the segment joining $x\in E_2$, $y\in E_2$, and $L(x, y)$ denotes the line joining $x$ and $y$ if $x\neq y$. (Also see Valentine [6].)

The main result in this section is contained in the following theorem.

THEOREM 1. *Suppose $B$ is a subset of a compact set $S$ in the plane $E_2$. Furthermore, suppose the following two conditions hold.*

(a) *For every three points $x$, $y$, $z$ in $B$ suppose there exists at least one point $p$ such that $xp$, $yp$, $zp$ all belong to $S$ (the point $p$ depends on $x$, $y$*

Received by the editors January 31, 1966.

[1] The preparation of this manuscript was sponsored by the National Science Foundation Grant 1988.

*and z, and x, y, z need not be distinct*).

(b) *For every four points* $x \in B$, $y \in B$, $p \in S$, $q \in S$ *the quadrilateral with consecutive edges* $xp$, $py$, $yq$, $qx$ *is the boundary of a bounded closed set which belongs to* $S$ (*the quadrilateral need not be simple*) *if the four edges* $xp$, $py$, $yq$, $qx$ *belong to* $S$.

*The above conditions are sufficient to imply that* $B$ *can see a point of* $S$ *via* $S$.

*Note.* It should be observed that if either $S$ is simply connected or if $S$ has the four-point simply-connected property of Definition 2, then (b) is also true, but not conversely.

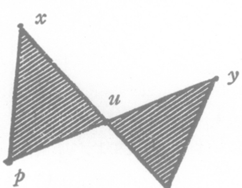
PROOF OF THEOREM 1. Instead of using Helly's theorem [**1**] to prove Theorem 1 (as was done in the proof of Krasnosel'skiĭ's theorem [**2**]) we will use a theorem of Molnar [**3**] in a manner similar to that of Robkin [**4**]. To do this, if $x \in B$ let $S(x)$ denote the set of all points of $S$ which can see $x$ via $S$, so that $S(x) \equiv \{y : xy \subset S\}$.

(1) Clearly $S(x)$ is a simply-connected compact set in $E_2$, since it is starshaped relative to $x$, and since $S$ is compact.

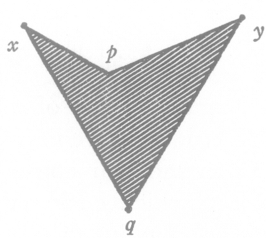
(2) Secondly, if $x \in B$, $y \in B$, then $S(x) \cap S(y)$ is a connected compact subset of $S$. To see this let

$$p \in S(x) \cap S(y), \qquad q \in S(x) \cap S(y).$$

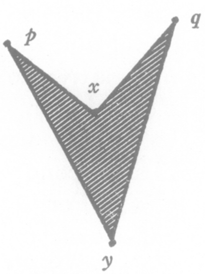
Clearly the quadrilateral $xpyq$ is essentially described in the following diagrams, in which $p$ and $q$ are interchangeable and $x$ and $y$ are interchangeable. In the first diagram, condition (b) implies $pu \cup uq \subset S(x)$



(1)

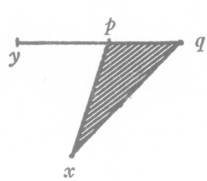


(2)



(3)

(4)



(5)

$\cap S(y)$ where $u = xq \cap yp$. (A corresponding fact holds if $p$ and $q$ are interchanged.) In the second diagram, condition (b) implies $pq \subset S(x) \cap S(y)$. In the third diagram, we have $qy \cup yp \subset S(x) \cap S(y)$. If $x$, $p$, $y$, $q$ are collinear, then $pq \subset S(x) \cap S(y)$. In the remaining case, illustrated in (5), we also have $pq \subset S(x) \cap S(y)$. In this manner, we have shown that $S(x) \cap S(y)$ is connected and compact if $x \in B$, $y \in B$.

(3) Finally, by hypothesis, we have $S(x) \cap S(y) \cap S(z) \neq 0$ for every three points $x$, $y$, $z$ in $B$.

To complete the proof we use the following theorem of Molnar.

THEOREM (MOLNAR, [**3**]). *A family of three or more simply connected compact sets in the plane $E_2$ has a nonempty simply connected intersection if every two of its members have a connected intersection and if every three of its members have a nonempty intersection.*

Since conditions (1), (2) and (3) above imply that the collection $\{S(x), x \in B\}$ satisfies the hypotheses of Molnar's theorem, we have

$$\bigcap_{x \in B} S(x) \neq 0$$

which implies that $B$ can see a point of $S$ via $S$.

It is of interest to observe that condition (a) alone is not sufficient to imply the conclusion of Theorem 1. For instance, let $S$ consist of that compact ring-shaped set bounded by two concentric circles, such that

$$S \equiv \{x: r \leqq \|x\| \leqq 2r\}$$

where $r > 0$. Let $B$ be the circle

$$B \equiv \{x: \|x\| = 2r\}.$$

It is quite clear that condition (a) of Theorem 1 holds for every three points $x$, $y$, $z$ in $B$. However, we may choose four points $x$, $y$, $p$, $q$ in $B$ (and hence in $S$) as the four consecutive vertices of a square whose edges lie in $S$ and which encloses the disk $C \equiv \{x: \|x\| \leqq r\}$. Hence, since $B$ cannot see a point of $S$ via $S$ we see that condition (a) is not sufficient. The ring-shaped set $S \equiv \{x: r \leqq \|x\| \leqq R\}$ and the circle $B \equiv \{x: \|x\| = R\}$ have an interesting relationship relative to properties (a) and (b) of Theorem 1. It is illuminating to examine this situation for different values of $r$ and $R$.

Finally, it would be desirable to characterize the sets $B$ since condition (b) is also not necessary for $B$ to see a point of $S$ via $S$. For instance, let $S$ be the boundary of a square, and let $B$ consist of two diagonally opposite vertices of $S$. Although $B$ can see a point of $S$ via $S$, and although condition (a) holds, condition (b) fails, so that (b) is not a necessary condition. The theory in $E_n$ for $n>2$ appears to be quite challenging, since the Helly topological theorems for $n>2$ are difficult to apply at the present moment.

2. In this section we characterize those bounded closed connected sets $S$ in $E_2$ which can be covered by a collection of parallel lines each member of which intersects $S$ in a connected set (a segment or a point). For related results see Robkin and Valentine [**5**].

THEOREM 2. *Let $S$ be a closed connected set in the Euclidean plane $E_2$. Suppose there exists a pair of points $a$ and $b$ in $S$ such that the following holds. For each pair of points $x$ and $y$ in the convex hull of $S$ there exists a pair of parallel lines, say $L(x)$ and $L(y)$, containing $x$ and $y$ respectively, each of which intersects the segment $ab$, and each of which intersects $S$ in a connected set ($L(x)$ and $L(y)$ do not have to be distinct).*

*Then $S$ can be covered by a family of parallel lines $\mathcal{P}$ each member of which intersects $S$ in a connected set. Furthermore, if $S$ is a bounded closed connected set then the converse holds.*

PROOF. If $S\subset L(a, b)$, then the theorem is clearly true, so we consider the case when $S \not\subset L(a, b)$. Choose $x\in\operatorname{conv} S$, and let $C(x)$ denote the union of all lines through $x$ which intersect $S$ in a connected set and which intersect $ab$. Without loss suppose the origin $\phi$ is not on the line $L(a, b)$, and let $D(x)$ be that translate of $C(x)$ so that $x$ goes to $\phi$; hence

$$D(x) \equiv C(x) - x.$$

Define $M(x)$ as follows,

$$M(x) \equiv \operatorname{conv} [L(a, b) \cap \operatorname{cl} D(x)],$$

and let

$$Q \equiv \{M(x) : x \in \operatorname{conv} S,\ x \notin L(a, b)\}.$$

Observe that $Q\neq 0$ since we are assuming $S \not\subset L(a, b)$.

By hypothesis, every two members $M(x)\in Q$, $M(y)\in Q$ have at least one point in common. Since each member of $Q$ is a compact interval (hence, also convex), Helly's theorem [**1**] for the line implies there exists a point

$$p \in \bigcap_{M(x)\in Q} M(x).$$

Let $L$ be the line in $E_2$ which is determined by the origin $\phi$ and the point $p$. Also if $x\in E_2$, let $L(x)$ denote the line through $x$ which is *parallel* to $L$. We will prove that the family of parallel lines $\mathcal{P}$ defined as follows

$$\mathcal{P} \equiv \{L(x),\, x \in S\}$$

has the properties stated in the theorem. To do this, suppose a point $y \in \operatorname{conv} S$ exists such that $L(y) \cap S$ is not connected. Then there exist points $u \in L(y) \cap S$, $v \in L(y) \cap S$ such that the relative interior of the segment $uv$ does not intersect $S$. Since $S$ is a closed connected set in $E_2$, the crosscut $uv$ of the complement of $S$ (as it is called) divides a component $K$ of the complement of $S$ so that at least one component of $K \sim uv$, say $K_1$, is bounded. (See Valentine, [**6**, pp. 201–202].) Choose a point $w \in K_1$, $w \notin L(a, b)$. Since $L(w) \cap S$ is not connected, let $cd$ be a crosscut of $K_1$ which is parallel to $uv$ and which contains $w$. Since $w \in K_1$, $K_1 \subset K \sim uv$, any line through $w$ which makes a sufficiently small angle with $L(w)$ must intersect $S$ in a disconnected set. Hence, since $L(w) \cap ab \subset \operatorname{conv}[ab \cap \operatorname{cl} C(w)]$, $L(w) \not\subset C(w)$, and since $C(w)$ is the union of all lines through $w$ which intersect $S$ in a connected set and which intersect $ab$, there exists, through $w$, two lines $L_i(w) \subset C(w)$ $(i=1, 2)$ such that $ab \cap L(w)$ lies between $ab \cap L_1(w)$ and $ab \cap L_2(w)$. Furthermore, since $L_i(w) \cap S$ $(i=1, 2)$ is connected, and since $w \notin S$, there exist rays $R_i(w) \subset L_i(w)$ $(i=1, 2)$ having $w$ as a common endpoint such that $S \cap R_i(w) = 0$. If $ab \cap R_i(w) \neq 0$ $(i=1, 2)$ then $R_1(w) \cup R_2(w)$ separates $c$ from $d$, violating the connectedness of $S$. Secondly, if $R_1(w) \cap ab = 0$ and $R_2(w) \cap ab \neq 0$, or if $R_1(w) \cap ab \neq 0$ and $R_2(w) \cap ab = 0$, then $R_1(w) \cup R_2(w)$ separates $a$ and $b$ in violation of the connectedness of $S$. Finally, if $R_i(w) \cap ab = 0$ $(i=1, 2)$, then $R_1(w) \cup R_2(w)$ separates $c$ and $d$, violating the connectedness of $S$. Since $R_1(w)$ and $R_2(w)$ must satisfy at least one of the above situations, we have a contradiction, so that $L(y) \cap S$ is connected for each point $y \in S$. Thus $\mathcal{P}$ has the property stated. When $S$ is also bounded, the converse follows trivially, so that Theorem 2 has been established.

## Bibliography

**1.** E. Helly, *Über Mengen konvexer Körper mit gemeinshaftlichen Punkten*, Jber. Deutsch. Math.-Verein. **32** (1923), 175–176.

**2.** M. A. Krasnosel'skiĭ, *Sur un critère pour qu'un domain soit étoilé*, Mat. Sb. (61) **19** (1946), 309–310.

**3.** J. Molnar, *Über den zweidimensionalen topologischen Satz von Helly*, Mat. Lapok **8** (1957), 108–114.

**4.** E. E. Robkin, *Characterizations of starshaped sets*, Doctoral Dissertation, Univ. of California, Los Angeles, 1965.

**5.** E. E. Robkin and F. A. Valentine, *Families of parallels associated with sets*, Pacific J. Math. **16** (1966), 147–157.

**6.** F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

University of California, Los Angeles