ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE WAVE EQUATIONS

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In the theory of scattering for hyperbolic equations, it is necessary to estimate the behavior of solutions to the unperturbed problem as well as the perturbed for large \(|t|\). At present most estimates for the wave equation or the relativistic wave equation are in the sup norm. (See [1]–[5].) The purpose of this paper is to present some simple but rather interesting estimates in \(L_2\) of solutions to

\[
\Box u = m^2 u, \quad m \geq 0,
\]

where

\[
\Box = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial t^2}.
\]

In particular, we will show that for finite energy solutions \(u\) of (1) \(\|u(x, t)\|_2\) has a definite limit depending on the initial data. It will follow that if \(\|u(x, t)\|_2\) tends to 0, \(u \equiv 0\). This seems to be a well-known "folk theorem."

1. The \(L_2\) norm. Let \(B = (m^2 - \Delta^2)^{1/2}\) considered as a linear operator on \(L_2(\mathbb{R}^n)\). If \(m > 0\), \(B\) has a bounded inverse. Let \(B(z) = (w^2 + z^2)^{1/2}\) where \(z = (z_1, \ldots, z_n)\) and \(z^2 = z_1^2 + \cdots + z_n^2\). We define the domain of \(B\) to be all \(f \in L_2\) such that \(B(z)F(z) \in L_2\) where \(F\) is the Fourier transform of \(f\). For suitable initial data, the following two integrals are constant.

\[
(2) \quad \Pi = \int_{\mathbb{R}^n} \left\{ \sum_{i=1}^{n} \left( \frac{\partial u(x, t)}{\partial x_i} \right)^2 + u_t^2 + m^2 u^2 \right\} dx = \int_{\mathbb{R}^n} \{ (Bu)^2 + u_t^2 \} dx,
\]

\[
(3) \quad \Gamma = \int_{\mathbb{R}^n} \{ u^2 + (B^{-1}u_t)^2 \} dx \quad \text{(if } m = 0, u_t(x, 0) \in \mathcal{D}_{B^{-1}}).\]

**Theorem 1.** Let \(u(x, 0) = f, u_t(x, 0) = g\).

\[
(1) \quad \lim_{|t| \to \infty} \int_{\mathbb{R}^n} \left\{ \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 + m^2 u \right\} dx = \frac{\Pi}{2},
\]

\[
(2) \quad \lim_{|t| \to \infty} \int_{\mathbb{R}^n} u^2(x, t) dx = \frac{\Gamma}{2}.
\]

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Proof. We will prove statement (2). (1) is similar. We first note that

\[ \Gamma = \Gamma(0) = \int_{\mathbb{R}^n} f^2 + (B^{-1}g)^2 \, dx \]

\[ = \int_{\mathbb{R}^n} |F|^2 + (m^2 + |\xi|^2)^{-1} |G|^2 \, d\xi \]

where \( F, G \) are the Fourier transforms of \( f, g \) respectively with respect to \( x \). By the functional calculus,

\[ U(z, t) = (\cos tB(z))F(z) + (B^{-1}(z) \sin tB(z))G(z). \]

But

\[ \int_{\mathbb{R}^n} u^2 \, dx = \int_{\mathbb{R}^n} |U|^2 \, dz \]

\[ = \int_{\mathbb{R}^n} \left[ \cos^2 tB |F|^2 + B^{-2}\sin^2 tB |G|^2 + B^{-1} \sin 2tB (\bar{F}\bar{G} + \bar{G}F) \right] \, dz \]

\[ = \frac{1}{2} \int_{\mathbb{R}^n} |F|^2 + B^{-2} |G|^2 \]

\[ + \int_{\mathbb{R}^n} \left[ \frac{3}{2} \cos 2tB (|F|^2 - |B^{-1}G|^2) + B^{-1} \sin 2tB (\bar{F}\bar{G} + \bar{G}F) \right] d\xi. \]

The theorem will be proved if we show the second integral tends to zero. However, this follows by a trivial modification of the Riemann-Lebesgue lemma.

Corollary. Let \( u \) be solution of \( \Box u = 0 \) with \( u(x, 0) \in L^2 \) and \( u_t(x, 0) \) in the domain of \( B^{-1} \). Then if \( \|u(\cdot, t)\|_{L^2} \) tends to zero as \( |t| \to \infty \), then \( u = 0 \) for all \( t \).

Proof. By the assumptions, \( \Gamma = 0 \). Thus \( u = 0 \).

Remark. Theorem 1 seems to suggest an equipartition or virial law of some kind for the energies.

References