1. Introduction. Let \( H_k \) be the space of polynomials of degree not greater than \( 2k+1 \), and let \( f = \hat{p} + w \), where \( \hat{p} \) is in \( H_k \) and \( w \) is a sample function from a stationary random process with zero mean and spectral density \( \psi(\theta) \). We study the problem of filtering \( f \) to obtain estimates of \( \hat{p} \) and its derivatives. Consider the polynomial weighting functions

\[
\begin{align*}
\psi_0(x) &= \sum_{i=0}^{2k+1} (j + \frac{1}{2}) P_j^{(2i)}(0) P_i(x), \\
\psi_i(x) &= \sum_{i=0}^{2k+1} (j + \frac{1}{2}) P_j^{(2i+1)}(0) P_i(x), \quad 0 \leq i \leq n,
\end{align*}
\]

where \( P_0, P_1, \ldots \) are the Legendre polynomials. Define

\[
g_i(x) = \int_{-1}^{1} u_{ik}(y)f(x + y)dy
\]

and

\[
h_i(x) = \int_{-1}^{1} v_{ik}(y)f(x + y)dy.
\]

The mean value of \( g_i(x) \) is given by

\[
E[g_i(x)] = \int_{-1}^{1} u_{ik}(y)\hat{p}(x + y)dy = \hat{p}^{(2i)}(x),
\]

and its random component is

\[
\mu_i(x) = \int_{-1}^{1} u_{ik}(y)w(x + y)dy.
\]

The second equality in (2) follows from the well-known reproducing property of the kernel

\[
K_k(x, y) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j(x) P_j(y)
\]

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in $H_k$. (More generally, it is well known that if $F$ is square integrable in $[-1, 1]$, then
\[ \int_{-1}^{1} K_k(x, y)F(y)dy \]
is the least squares polynomial fit to $F$ of degree not greater than $2k+1$.) It can be shown that $r_i(x)$ is a sample function from a stationary random process with zero mean and spectral density

\[ \alpha_i(\theta) = C_{ik}(\theta)\psi(\theta), \]
where

\[ C_{ik}(\theta) = \int_{-1}^{1} u_{ik}(x) \cos x\theta dx. \]

It is also known that if the process $w$ is white noise, (that is, if $\psi = \text{constant}$), then $g_i(x)$ is the minimum variance estimate of $\rho^{(2i)}(x)$. Similarly,

\[ E[h_i(x)] = \rho^{(2i+1)}(x), \]
and the error

\[ \rho^{(2i+1)}(x) - h_i(x) \]
has zero mean and spectral density

\[ \beta_i(\theta) = S_{ik}(\theta)\psi(\theta), \]
where

\[ S_{ik}(\theta) = \int_{-1}^{1} v_{ik}(x) \sin x\theta dx. \]

The magnitudes of $C_{ik}$ and $S_{ik}$ are of interest because of (3) and (6). In [1], the author established the following theorem.

**Theorem 1.** For each $k \geq 0$, $C_{0k}(0) = 1$, and $|C_{0k}(\theta)| < 1$ if $\theta \neq 0$.

In this paper we prove the following theorem.

**Theorem 2.** Let $k \geq 0$, and $0 \leq i \leq k$. Then

\[ |\theta^{-2i}C_{ik}(\theta)| < 1, \]

and

\[ |\theta^{-(2i+1)}S_{ik}(\theta)| < 1, \]
if \( \theta \neq 0 \), and

\[
\lim_{\theta \to 0} \theta^{-2i} C_{ik}(\theta) = \lim_{\theta \to 0} \theta^{-(2i+1)} S_{ik}(\theta) = (-1)^i.
\]

2. **Proof of Theorem 2.** The proof uses Theorem 1 and an induction argument. We need certain lemmas.

**Lemma 1.** Let

\[
F_i(\theta) = \int_{-1}^{1} x^{2i} \cos x \theta dx,
\]

\[
G_i(\theta) = \int_{-1}^{1} x^{2i+1} \sin x \theta dx,
\]

and

\[
\sigma_i = \int_{-1}^{1} x^{2i} dx = (i + \frac{1}{2})^{-1}.
\]

Then

\[
F_i(\theta) = \sum_{j=0}^{k} \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j)!}, \quad 0 \leq i \leq k,
\]

and

\[
G_i(\theta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{S_{jk}(\theta)}{(2j + 1)!}, \quad 0 \leq i \leq k.
\]

**Proof.** Let

\[
f_k(x, \theta) = \int_{-1}^{1} K_k(x, y) \cos y \theta dy.
\]

Since \( f_k \) is the projection of \( \cos x \theta \) on \( H_k \),

\[
\int_{-1}^{1} [\cos y \theta - f_k(y, \theta)] y^{2i} dy = 0, \quad 0 \leq i \leq k.
\]

By expanding \( K_k \) in powers of \( x \), we find that

\[
f_k(x, \theta) = \sum_{j=0}^{k} \frac{C_{jk}(\theta)x^{2j}}{(2j)!},
\]

which, when substituted into (14), yields (12). Equation (13) can be found similarly, by considering the projection of \( \sin x \theta \) on \( H_k \).
Lemma 2. If $C_{kk}(\alpha) = 0$, then $C_{ik}(\alpha) = C_{i,k-1}(\alpha)$, $0 \leq i \leq k-1$, and if $S_{kk}(\beta) = 0$, then $S_{ik}(\beta) = S_{i,k-1}(\beta)$, $0 \leq i \leq k-1$.

Proof. If the highest degree term in $f_k$ vanishes, then $f_k = f_{k-1}$, from elementary properties of least squares polynomials. This implies the first part, and the second is obtained similarly.

Lemma 3. Let

$$\alpha_{ik} = \left( (-1)^i \prod_{q=0}^{k} \left( i + q + \frac{1}{2} \right) \right) / i!(k-i)!$$

and

$$\beta_{ik} = \left( (-1)^i \prod_{q=0}^{k} \left( i + q + \frac{3}{2} \right) \right) / i!(k-i)!.$$ 

Then

$$C_{ik}(\theta) = (2i)! \alpha_{ik} \sum_{j=0}^{k} \alpha_{jk} \sigma_{i+j} F_j(\theta),$$

and

$$S_{ik}(\theta) = (2i+1)! \beta_{ik} \sum_{j=0}^{k} \beta_{jk} \sigma_{i+j+1} G_j(\theta).$$

Proof. The matrices of the systems (12) and (13) have general elements of the form $(a_i + b_j)^{-1}$. Using the method of [2, Vol. 2, pp. 98, 299], it can be shown that their inverses have $i,j$th elements given by $\alpha_{ik} \alpha_{jk} \sigma_{i+j}$ and $\beta_{ik} \beta_{jk} \sigma_{i+j+1}$, respectively, where $0 \leq i, j \leq k$.

Lemma 4.

$$\frac{d}{d\theta} \left\{ \theta^{-2i-1} S_{i,k-1}(\theta) \right\} = - \frac{\beta_{i,k-1}(2i+1)!}{\alpha_{kk}(2k)!} C_{kk}(\theta) \theta^{-2i-1}, \quad 0 \leq i \leq k-1,$$

(16)

and

$$\frac{d}{d\theta} \left\{ \theta^{-2i} C_{ik}(\theta) \right\} = - \frac{\alpha_{ik}(2i)!}{\beta_{kk}(2k+1)!} S_{kk}(\theta) \theta^{-2i}, \quad 0 \leq i \leq k.$$ 

(17)

Proof. To establish (16), we start by dividing (15), with $k$ replaced by $k-1$, by $\theta^{i+1}$, and differentiating. Note that
\[
\frac{d}{d\theta} \{G_j(\theta)\theta^{-2i-1}\} = G'_j(\theta)\theta^{-2i-1} - (2i + 1)G_j(\theta)\theta^{-2i-2}
\]

\[
= \theta^{-2i-1}\left(\frac{2i + 2j + 3}{2j + 2} F_{j+1}(\theta) - \frac{2i + 1}{j + 1} \sin \theta\right),
\]

where the last equality follows from the relations

(18) \[G'_j(\theta) = F_{j+1}(\theta),\]

and

\[G_j(\theta) = (\sin \theta)/(j + 1) - (\theta F_{j+1}(\theta))/(2j + 2),\]

which can be deduced from (10) and (11). Using the fact that

\[F_0(\theta) = (2 \sin \theta)/\theta,\]

it can now be seen that

\[
\frac{d}{d\theta} \left(\theta^{-2i-1}S_{i,k-1}(\theta)\right)
\]

\[= \left(\frac{2i + 1)!}{\theta^{2i+1}} \sum_{j=0}^{k} \frac{\beta_{j,k-1}F_{j+1}(\theta)}{j + 1} - \frac{F_0(\theta)}{\sigma_i} \sum_{j=0}^{k-1} \frac{\beta_{j,k-1}\sigma_{i+j+1}}{j + 1}\right).\]

It is easy to verify that

\[\beta_{j,k-1}/(j + 1) = -\alpha_{j+1,k}\sigma_{j+k+1}\]

and

\[\sum_{j=0}^{k} \alpha_{jk}\sigma_{i+j+1} = 0, \quad 0 \leq i \leq k - 1.\]

Using these and (19), (16) can be established. Equation (17) can be derived similarly, using the relations

\[F'_j(\theta) = -G_j(\theta), \quad F_j(\theta) = \sigma_j \left(\cos \theta + \frac{\theta}{2} G_j(\theta)\right),\]

(20) \[\alpha_{jk}\sigma_j = \beta_{jk}\sigma_{k+j+1},\]

and

\[\sum_{i=0}^{k} \alpha_{jk}\sigma_{i+j}\sigma_j = 0, \quad 1 \leq i \leq k.\]

**Lemma 5.** The functions \(\theta^{-2i}C_{ik}(\theta), \ 0 \leq i \leq k,\) have the same critical points. Also, the functions \(\theta^{-2i-1}S_{ik}(\theta), \ 0 \leq i \leq k,\) have the same critical
points. Furthermore, if \( \phi \) is a critical point of the former, then
\[
-C_{ik}(\phi) / \phi = S_{i-1,k-1}(\phi), \quad 1 \leq i \leq k,
\]
and if \( \eta \) is a critical point of the latter, then
\[
\frac{S_{ik}(\eta)}{\eta} = C_{ik}(\eta), \quad 0 \leq i \leq k.
\]

**Proof.** From Lemma 4, the critical points of the first set of functions are the zeroes of \( S_{kk} \), while the critical points of the second set are the zeroes of \( C_{k+1,k+1} \). This implies the first two statements. To prove (21), multiply and divide the \( j \)th term on the right of (12) by \( \theta^{2i} \) and differentiate, to obtain
\[
F'_i(\theta) = \sum_{j=1}^{k} \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j - 1)!!} + \sum_{j=0}^{k} \sigma_{i+j} \frac{\theta^{2j}}{(2j)!} \frac{d}{d\theta} \left( \theta^{-2i} C_{jk}(\theta) \right).
\]
Let \( \theta = \phi \), where \( S_{kk}(\phi) = 0 \), so that the second sum vanishes. Making use of (20), we can write
\[
G_i(\phi) = -\sum_{j=0}^{k-1} \sigma_{i+j+1} \frac{C_{j+1,k}(\phi)}{(2j + 1)!!} \phi, \quad 0 \leq i \leq k.
\]
Since \( S_{kk}(\phi) = 0 \), we can infer (21) by comparing this system with (13) (with \( \theta = \phi \)), noting that the solution of the latter is unique, and using Lemma 2.

To derive (22), let \( C_{k+1,k+1}(\eta) = 0 \). Use (12) with \( k \) replaced by \( k+1 \) and \( \theta = \eta \), and Lemma 2, to conclude that
\[
F_i(\eta) = \sum_{j=0}^{k} \sigma_{i+j} \frac{C_{jk}(\eta)}{(2j)!}, \quad 0 \leq i \leq k+1.
\]
The last \( k \) of these equations can be written
\[
F_{i+1}(\eta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{C_{jk}(\eta)}{(2j)!}, \quad 0 \leq i \leq k.
\]
Now multiply and divide the \( j \)th term on the right side of (13) by \( \theta^{2i+1} \), differentiate as in the derivation of (23), set \( \theta = \eta \), and use (18). The result is
\[
F_{i+1}(\eta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{S_{jk}(\eta)}{(2j)! \eta}, \quad 0 \leq i \leq k.
\]
Comparing this with (24) yields (22).
We can now complete the proof of Theorem 2, by induction. Assume that \( k \geq 1 \), and that Theorem 2 holds for \( k - 1 \). This is so if \( k = 1 \), from Theorem 1. Let \( \phi \) be a nonzero critical point of \( \theta^{-2i}C_{ik}(\theta) \), and divide both sides of (21) by \( \phi^{2i-1} \) to obtain

\[-\phi^{-2i}C_{ik}(\phi) = \phi^{-2i+1}S_{i-1,k-1}(\phi), \quad 1 \leq i \leq k.\]

From the induction assumption, it now follows that

\[|\theta^{-2i}C_{ik}(\theta)| < 1, \quad 1 \leq i \leq k,\]

if \( \theta = \phi \), a nonzero critical point of \( \theta^{-2i}C_{ik}(\theta) \), and therefore the inequality holds for all \( \theta \neq 0 \). (The inequality with \( i = 0 \) does not follow from this argument, but it has already been established in Theorem 1.) Now let \( \eta \) be a nonzero critical point of \( \theta^{-2i-1}S_{ik}(\theta) \), and divide both sides of (22) by \( \eta^{2i} \) to find that

\[\eta^{-2i-1}S_{ik}(\eta) = \eta^{-2i}C_{ik}(\eta), \quad 0 \leq i \leq k.\]

We have just established that the right side is less than unity in magnitude. Hence (8) holds for \( \theta = \eta \), and therefore for any \( \theta \neq 0 \).

The limits (9) are obtained by expanding (4) and (7) in powers of \( \theta \), integrating term by term, and noting that

\[\int_{-1}^{1} u_{ik}(x)x^{2i}dx = (2i)!(\delta_{ij}),\]

\[\int_{-1}^{1} v_{ik}(x)x^{2i+1}dx = (2i + 1)!(\delta_{ij}), \quad 0 \leq i, j \leq k.\]

References


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