

# BOUNDS ON THE GENERATING FUNCTIONS OF CERTAIN SMOOTHING OPERATIONS

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1. **Introduction.** Let  $H_k$  be the space of polynomials of degree not greater than  $2k+1$ , and let  $f=p+w$ , where  $p$  is in  $H_k$  and  $w$  is a sample function from a stationary random process with zero mean and spectral density  $\psi(\theta)$ . We study the problem of filtering  $f$  to obtain estimates of  $p$  and its derivatives. Consider the polynomial weighting functions

$$u_{ik}(x) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j^{(2i)}(0) P_j(x),$$

$$v_{ik}(x) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j^{(2i+1)}(0) P_j(x), \quad 0 \leq i \leq n,$$

where  $P_0, P_1, \dots$  are the Legendre polynomials. Define

$$g_i(x) = \int_{-1}^1 u_{ik}(y) f(x+y) dy$$

and

$$(1) \quad h_i(x) = \int_{-1}^1 v_{ik}(y) f(x+y) dy.$$

The mean value of  $g_i(x)$  is given by

$$(2) \quad E[g_i(x)] = \int_{-1}^1 u_{ik}(y) p(x+y) dy = p^{(2i)}(x),$$

and its random component is

$$\mu_i(x) = \int_{-1}^1 u_{ik}(y) w(x+y) dy.$$

The second equality in (2) follows from the well-known reproducing property of the kernel

$$K_k(x, y) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j(x) P_j(y)$$

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in  $H_k$ . (More generally, it is well known that if  $F$  is square integrable in  $[-1, 1]$ , then

$$\int_{-1}^1 K_k(x, y)F(y)dy$$

is the least squares polynomial fit to  $F$  of degree not greater than  $2k+1$ .) It can be shown that  $r_i(x)$  is a sample function from a stationary random process with zero mean and spectral density

$$(3) \quad \alpha_i(\theta) = C_{ik}^2(\theta)\psi(\theta),$$

where

$$(4) \quad C_{ik}(\theta) = \int_{-1}^1 u_{ik}(x) \cos x\theta dx.$$

It is also known that if the process  $w$  is white noise, (that is, if  $\psi = \text{constant}$ ), then  $g_i(x)$  is the minimum variance estimate of  $p^{(2i)}(x)$ . Similarly,

$$(5) \quad E[h_i(x)] = p^{(2i+1)}(x),$$

and the error

$$p^{(2i+1)}(x) - h_i(x)$$

has zero mean and spectral density

$$(6) \quad \beta_i(\theta) = S_{ik}^2(\theta)\psi(\theta),$$

where

$$(7) \quad S_{ik}(\theta) = \int_{-1}^1 v_{ik}(x) \sin x\theta dx.$$

The magnitudes of  $C_{ik}$  and  $S_{ik}$  are of interest because of (3) and (6). In [1], the author established the following theorem.

**THEOREM 1.** *For each  $k \geq 0$ ,  $C_{0k}(0) = 1$ , and  $|C_{0k}(\theta)| < 1$  if  $\theta \neq 0$ .*

In this paper we prove the following theorem.

**THEOREM 2.** *Let  $k \geq 0$ , and  $0 \leq i \leq k$ . Then*

$$|\theta^{-2i}C_{ik}(\theta)| < 1,$$

and

$$(8) \quad |\theta^{-(2i+1)}S_{ik}(\theta)| < 1,$$

if  $\theta \neq 0$ , and

$$(9) \quad \lim_{\theta \rightarrow 0} \theta^{-2i} C_{ik}(\theta) = \lim_{\theta \rightarrow 0} \theta^{-(2i+1)} S_{ik}(\theta) = (-1)^i.$$

2. **Proof of Theorem 2.** The proof uses Theorem 1 and an induction argument. We need certain lemmas.

LEMMA 1. *Let*

$$(10) \quad F_i(\theta) = \int_{-1}^1 x^{2i} \cos x\theta dx,$$

$$(11) \quad G_i(\theta) = \int_{-1}^1 x^{2i+1} \sin x\theta dx,$$

and

$$\sigma_i = \int_{-1}^1 x^{2i} dx = (i + \frac{1}{2})^{-1}.$$

Then

$$(12) \quad F_i(\theta) = \sum_{j=0}^k \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j)!}, \quad 0 \leq i \leq k,$$

and

$$(13) \quad G_i(\theta) = \sum_{j=0}^k \sigma_{i+j+1} \frac{S_{jk}(\theta)}{(2j+1)!}, \quad 0 \leq i \leq k.$$

PROOF. Let

$$f_k(x, \theta) = \int_{-1}^1 K_k(x, y) \cos y\theta dy.$$

Since  $f_k$  is the projection of  $\cos x\theta$  on  $H_k$ ,

$$(14) \quad \int_{-1}^1 [\cos y\theta - f_k(y, \theta)] y^{2i} dy = 0, \quad 0 \leq i \leq k.$$

By expanding  $K_k$  in powers of  $x$ , we find that

$$f_k(x, \theta) = \sum_{j=0}^k \frac{C_{jk}(\theta) x^{2j}}{(2j)!},$$

which, when substituted into (14), yields (12). Equation (13) can be found similarly, by considering the projection of  $\sin x\theta$  on  $H_k$ .

LEMMA 2. If  $C_{kk}(\alpha) = 0$ , then  $C_{ik}(\alpha) = C_{i,k-1}(\alpha)$ ,  $0 \leq i \leq k-1$ , and if  $S_{kk}(\beta) = 0$ , then  $S_{ik}(\beta) = S_{i,k-1}(\beta)$ ,  $0 \leq i \leq k-1$ .

PROOF. If the highest degree term in  $f_k$  vanishes, then  $f_k = f_{k-1}$ , from elementary properties of least squares polynomials. This implies the first part, and the second is obtained similarly.

LEMMA 3. Let

$$\alpha_{ik} = \left( (-1)^i \prod_{q=0}^k (i + q + \frac{1}{2}) \right) / i!(k-i)!$$

and

$$\beta_{ik} = \left( (-1)^i \prod_{q=0}^k (i + q + \frac{3}{2}) \right) / i!(k-i)!$$

Then

$$C_{ik}(\theta) = (2i)! \alpha_{ik} \sum_{j=0}^k \alpha_{jk} \sigma_{i+j} F_j(\theta),$$

and

$$(15) \quad S_{ik}(\theta) = (2i+1)! \beta_{ik} \sum_{j=0}^k \beta_{jk} \sigma_{i+j+1} G_j(\theta).$$

PROOF. The matrices of the systems (12) and (13) have general elements of the form  $(a_i + b_j)^{-1}$ . Using the method of [2, Vol. 2, pp. 98, 299], it can be shown that their inverses have  $i, j$ th elements given by  $\alpha_{ik} \alpha_{jk} \sigma_{i+j}$  and  $\beta_{ik} \beta_{jk} \sigma_{i+j+1}$ , respectively, where  $0 \leq i, j \leq k$ .

LEMMA 4.

$$(16) \quad \frac{d}{d\theta} \{ \theta^{-2i-1} S_{i,k-1}(\theta) \} = - \frac{\beta_{i,k-1} (2i+1)!}{\alpha_{kk} (2k)!} C_{kk}(\theta) \theta^{-2i-1},$$

$$0 \leq i \leq k-1,$$

and

$$(17) \quad \frac{d}{d\theta} \{ \theta^{-2i} C_{ik}(\theta) \} = - \frac{\alpha_{ik} (2i)!}{\beta_{kk} (2k+1)!} S_{kk}(\theta) \theta^{-2i}, \quad 0 \leq i \leq k.$$

PROOF. To establish (16), we start by dividing (15), with  $k$  replaced by  $k-1$ , by  $\theta^{2i+1}$ , and differentiating. Note that

$$\begin{aligned} \frac{d}{d\theta} \{G_j(\theta)\theta^{-2i-1}\} &= G'_j(\theta)\theta^{-2i-1} - (2i+1)G_j(\theta)\theta^{-2i-2} \\ &= \theta^{-2i-1} \left( \frac{2i+2j+3}{2j+2} F_{j+1}(\theta) - \frac{2i+1}{j+1} \frac{\sin \theta}{\theta} \right), \end{aligned}$$

where the last equality follows from the relations

$$(18) \quad G'_j(\theta) = F_{j+1}(\theta),$$

and

$$G_j(\theta) = (\sin \theta)/(j+1) - (\theta F_{j+1}(\theta))/(2j+2),$$

which can be deduced from (10) and (11). Using the fact that

$$F_0(\theta) = (2 \sin \theta)/\theta,$$

it can now be seen that

$$(19) \quad \begin{aligned} &\frac{d}{d\theta} (\theta^{-2i-1} S_{i,k-1}(\theta)) \\ &= \frac{(2i+1)! \beta_{i,k-1}}{\theta^{2i+1}} \left\{ \sum_{j=0}^{k-1} \frac{\beta_{j,k-1} F_{j+1}(\theta)}{j+1} - \frac{F_0(\theta)}{\sigma_i} \sum_{j=0}^{k-1} \frac{\beta_{j,k-1} \sigma_{i+j+1}}{j+1} \right\}. \end{aligned}$$

It is easy to verify that

$$\beta_{j,k-1}/(j+1) = -\alpha_{j+1,k} \sigma_{j+k+1}$$

and

$$\sum_{j=0}^k \alpha_{jk} \sigma_{j+k} \sigma_{i+j} = 0, \quad 0 \leq i \leq k-1.$$

Using these and (19), (16) can be established. Equation (17) can be derived similarly, using the relations

$$(20) \quad \begin{aligned} F'_j(\theta) &= -G_j(\theta), \quad F_j(\theta) = \sigma_j \left( \cos \theta + \frac{\theta}{2} G_j(\theta) \right), \\ &\alpha_{jk} \sigma_j = \beta_{jk} \sigma_{k+j+1}, \end{aligned}$$

and

$$\sum_{j=0}^k \alpha_{jk} \sigma_{i+j} \sigma_j = 0, \quad 1 \leq i \leq k.$$

LEMMA 5. *The functions  $\theta^{-2i} C_{ik}(\theta)$ ,  $0 \leq i \leq k$ , have the same critical points. Also, the functions  $\theta^{-2i-1} S_{ik}(\theta)$ ,  $0 \leq i \leq k$ , have the same critical*

points. Furthermore, if  $\phi$  is a critical point of the former, then

$$(21) \quad -C_{ik}(\phi)/\phi = S_{i-1,k-1}(\phi), \quad 1 \leq i \leq k,$$

and if  $\eta$  is a critical point of the latter, then

$$(22) \quad \frac{S_{ik}(\eta)}{\eta} = C_{ik}(\eta), \quad 0 \leq i \leq k.$$

PROOF. From Lemma 4, the critical points of the first set of functions are the zeroes of  $S_{kk}$ , while the critical points of the second set are the zeroes of  $C_{k+1,k+1}$ . This implies the first two statements. To prove (21), multiply and divide the  $j$ th term on the right of (12) by  $\theta^{2j}$  and differentiate, to obtain

$$(23) \quad F'_i(\theta) = \sum_{j=1}^k \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j-1)\theta} + \sum_{j=0}^k \sigma_{i+j} \frac{\theta^{2j}}{(2j)!} \frac{d}{d\theta} (\theta^{-2j} C_{jk}(\theta)).$$

Let  $\theta = \phi$ , where  $S_{kk}(\phi) = 0$ , so that the second sum vanishes. Making use of (20), we can write

$$G_i(\phi) = - \sum_{j=0}^{k-1} \sigma_{i+j+1} \frac{C_{j+1,k}(\phi)}{(2j+1)! \phi}, \quad 0 \leq i \leq k.$$

Since  $S_{kk}(\phi) = 0$ , we can infer (21) by comparing this system with (13) (with  $\theta = \phi$ ), noting that the solution of the latter is unique, and using Lemma 2.

To derive (22), let  $C_{k+1,k+1}(\eta) = 0$ . Use (12) with  $k$  replaced by  $k+1$  and  $\theta = \eta$ , and Lemma 2, to conclude that

$$F_i(\eta) = \sum_{j=0}^k \sigma_{i+j} \frac{C_{jk}(\eta)}{(2j)!}, \quad 0 \leq i \leq k+1.$$

The last  $k$  of these equations can be written

$$(24) \quad F_{i+1}(\eta) = \sum_{j=0}^k \sigma_{i+j+1} \frac{C_{jk}(\eta)}{(2j)!}, \quad 0 \leq i \leq k.$$

Now multiply and divide the  $j$ th term on the right side of (13) by  $\theta^{2j+1}$ , differentiate as in the derivation of (23), set  $\theta = \eta$ , and use (18). The result is

$$F_{i+1}(\eta) = \sum_{j=0}^k \sigma_{i+j+1} \frac{S_{jk}(\eta)}{(2j)! \eta}, \quad 0 \leq i \leq k.$$

Comparing this with (24) yields (22).

We can now complete the proof of Theorem 2, by induction. Assume that  $k \geq 1$ , and that Theorem 2 holds for  $k-1$ . This is so if  $k=1$ , from Theorem 1. Let  $\phi$  be a nonzero critical point of  $\theta^{-2i}C_{ik}(\theta)$ , and divide both sides of (21) by  $\phi^{2i-1}$  to obtain

$$-\phi^{-2i}C_{ik}(\phi) = \phi^{-2i+1}S_{i-1,k-1}(\phi), \quad 1 \leq i \leq k.$$

From the induction assumption, it now follows that

$$|\theta^{-2i}C_{ik}(\theta)| < 1, \quad 1 \leq i \leq k,$$

if  $\theta = \phi$ , a nonzero critical point of  $\theta^{-2i}C_{ik}(\theta)$ , and therefore the inequality holds for all  $\theta \neq 0$ . (The inequality with  $i=0$  does not follow from this argument, but it has already been established in Theorem 1.) Now let  $\eta$  be a nonzero critical point of  $\theta^{-2i-1}S_{ik}(\theta)$ , and divide both sides of (22) by  $\eta^{2i}$  to find that

$$\eta^{-2i-1}S_{ik}(\eta) = \eta^{-2i}C_{ik}(\eta), \quad 0 \leq i \leq k.$$

We have just established that the right side is less than unity in magnitude. Hence (8) holds for  $\theta = \eta$ , and therefore for any  $\theta \neq 0$ .

The limits (9) are obtained by expanding (4) and (7) in powers of  $\theta$ , integrating term by term, and noting that

$$\int_{-1}^1 u_{ik}(x)x^{2j}dx = (2i)! \delta_{ij},$$

$$\int_{-1}^1 v_{ik}(x)x^{2j+1}dx = (2i+1)! \delta_{ij}, \quad 0 \leq i, j \leq k.$$

#### REFERENCES

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2. G. Polya and G. Szego, *Aufgaben und Lehrsätze aus der Analysis*, Springer, Berlin, 1925.

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