CREATIVE AND WEAKLY CREATIVE SEQUENCES
OF r.e. SETS

V. D. VUCKOVIC

1. In [1] Cleave introduced the notion of a creative sequence of r.e. (recursively enumerable) sets and proved that all such sequences are r. (recursively) isomorphic and 1-1 universal for the class of all r.e. sequences of r.e. sets. In [2] and [3] Lachlan introduced an alternate definition and proved its equivalency with the definition of Cleave.

A sequence of r.e. sets $E_0, E_1, \ldots$ is called r.e. iff there is an r. function $g$ such that $E_i = w_{g(i)}$ for every $i \in \mathbb{N}$, where

$$x \in w_i \leftrightarrow \forall \, y \exists \, n \phi_i(n, x, y).$$

Cleave calls a disjoint r.e. sequence $E_0, E_1, \ldots$ of r.e. sets creative if there is a p. (partial) r. function $f$ such that for every disjoint r.e. sequence $w_{h(i)}$, $i=0, 1, \ldots$, (with recursive $h$) satisfying $E_i \cap w_{h(i)} = \emptyset$, for all $i$, we have, for every $x \in I(h)$,

$$f(x) \in \bigcup_{\mu = 0}^{\infty} (w_{h(\mu)} \cup E_\mu).$$

$I(h)$ is the set of indices of $h$ in the standard enumeration

$$\phi_0, \phi_1, \phi_2, \ldots,$$

of all r.p. functions, i.e.,

$$\phi_i(x) \simeq U(\mu_\nu T_1(i, x, y)).$$

Lachlan, in [2], proceeds as follows. Let first $g$ be recursive and such that

$$\forall \, n \exists \, x \forall \, y \phi_2(i, n, x, y) \leftrightarrow \forall \, y \phi_1(i, n, x, y).$$

Define the double sequence $W_{i,n}$ of r.e. sets by $W_{i,n} = w_{g(i,n)}$.

After Lachlan, an r.e. sequence $E_0, E_1, \ldots$ of r.e. sets is creative iff there is a recursive $f$ such that for all $i$

$$W_{i,f(i)} \cup E_{f(i)} \subset \bigcup_{\mu = 0}^{\infty} (W_{i,\mu} \cap E_\mu).$$

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Both Cleave's and Lachlan's definition seem to demand very much to be satisfied: (1.1) involves all indices $x$ of $h$, and (1.5) all indices $i$ (which are, in essence, indices of all r.e. sequences).

In this paper we propose a very weak definition of a creative sequence and prove its equivalency with the definition of Cleave (and so with the definition of Lachlan). Moreover, our definition is a direct generalization of the corresponding Smullyan's definition of a doubly weakly creative pair (Smullyan [4, p. 114]).

2. Obviously, a sequence $A_0, A_1, \cdots$ of r.e. sets is r.e. iff the predicate $x \in A_y$ is r.e. Let $\gamma$ be recursive and such that

\[ (2.1) \quad \forall T_2(u, \mu, x, y) \leftrightarrow \forall T_1(\gamma(\mu, u), x, y). \]

For every r.e. predicate $Q(\mu, x)$ there is an $e \in \mathbb{N}$ such that $Q(\mu, x) \leftrightarrow V T_2(e, \mu, x, y)$. With $Q(\mu, x) \leftrightarrow x \subseteq A_\mu$ we conclude: every r.e. sequence of r.e. sets can be represented as a sequence $w_{\gamma(\mu, e)}$ $\mu = 0, 1, \cdots$ for some $e$.

By the recursion theorem, for every r.e. predicate $Q(\mu, z, x, u)$ there is a recursive $\phi$ such that for all $i \in \mathbb{N}$,

\[ (2.2) \quad Q(\mu, i, x, \phi(i)) \leftrightarrow \forall T_2(\phi(i), \mu, x, y) \]

i.e., by (2.1),

\[ (2.3) \quad Q(\mu, i, x, \phi(i)) \leftrightarrow \forall T_1(\gamma(\mu, \phi(i)), x, y). \]

**Lemma 2.1.** Let $A_0, A_1, \cdots$ be an r.e. sequence of r.e. sets and let $f$ be any r. function. Then there is an r. function $\phi$ such that, for every $i \in \mathbb{N}$,

\[ (2.4) \quad i \in A_\mu \rightarrow w_{\gamma(\mu, \phi(i))} = \{f(\phi(i))\}; \]

\[ (2.5) \quad i \notin A_\mu \rightarrow w_{\gamma(\mu, \phi(i))} = \emptyset. \]

**Proof.** ($\{a\}$ denotes the singleton whose unique element is $a$; $\emptyset$ is the empty set.) In (2.3) take

\[ Q(\mu, z, x, u) \leftrightarrow z \in A_\mu \land x = f(u). \]

From this lemma we obtain immediately.

**Lemma 2.2.** Let $A_0, A_1, \cdots$ be a disjoint r.e. sequence of r.e. sets. Then there is an r. function $\phi$ such that, for every $i \in \mathbb{N}$,

\[ (2.6) \quad i \in A_\mu \rightarrow w_{\gamma(\mu, \phi(i))} = \{f(\phi(i))\} \quad \text{and all others } w_{\gamma(\nu, \phi(i))} \text{ are empty for } \nu \neq \mu, \text{ and} \]

\[ (2.7) \quad i \notin A_\mu \rightarrow \text{all } w_{\gamma(\mu, \phi(i))} \text{ are empty}. \]
**Definition 2.1.** An r.e. sequence $A_0, A_1, \ldots$ of r.e. sets is **meager** iff either all $A_\mu$ are empty or all but one are empty and this one, which is not empty, is a singleton.

**Definition 2.2.** A disjoint r.e. sequence $A_0, A_1, \ldots$ of r.e. sets is **weakly creative** under an r. function $f$ iff, for all $i \in \mathbb{N}$ for which the sequence $w_{\gamma(0,i)}, w_{\gamma(1,i)}, \ldots$ is meager,

(a) in case all $w_{\gamma(\mu,i)}$ are empty we have

$$f(i) \in \bigcup_{\mu=0}^{\infty} A_{\mu};$$

(b) in case $w_{\gamma(n_1,i)}$ is not empty and $w_{\gamma(n_0,i)} \cap A_{n_0} = \emptyset$, we have

$$f(i) \in w_{\gamma(n_0,i)}.$$

**3.** We prove some theorems from which will follow the equivalency of the weak creativity and the creativity in the sense of Cleave.

**Theorem 3.1.** If the sequence $E = E_0, E_1, \ldots$ is weakly creative then every disjoint r.e. sequence $A = A_0, A_1, \ldots$ of r.e. sets is reducible to $E$.

**Proof.** Let $E$ be creative under $f$. By Lemma 2.2 there is an r. function $\phi$ such that for every sequence $\Omega_i = w_{\gamma(0,\phi(i))}, w_{\gamma(1,\phi(i))}, \ldots$, we have

(3.1) $i \in A_\mu \rightarrow \Omega_i$ is meager and $w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}$, and

(3.2) $i \in A_\mu \rightarrow \Omega_i$ is meager and all $w_{\gamma(\mu,\phi(i))}$ are empty.

We shall prove that $\psi = f(\phi)$ reduces $A$ to $E$.

Suppose first that $i \in A_\mu$. Then $w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}$ and, therefore,

$$f(\phi(i)) \in w_{\gamma(\mu,\phi(i))}.$$  

If now $w_{\gamma(\mu,\phi(i))} \cap E_\mu = \emptyset$ we will have, by (2.9), $f(\phi(i)) \in w_{\gamma(\mu,\phi(i))}$ in contradiction to (3.3). Therefore, $f(\phi(i)) \in E_\mu$.

To prove the opposite inclusion

$$f(\phi(i)) \in E_\mu \rightarrow i \in A_\mu$$

suppose, contrary, that there is a $q \in \mathbb{N}$ such that $f(\phi(i)) \in E_q$ but $i \notin A_q$.

Now, if $i \in \bigcup_{\mu=0}^{\infty} A_\mu$, $\Omega_i$ consists of empty sets only, and (2.8) gives $f(\phi(i)) \in \bigcup_{\mu=0}^{\infty} E_\mu$—a contradiction. So, there is an $s \in \mathbb{N}$ such that $i \in A_s$. By the first part of the proof we obtain $f(\phi(i)) \in E_s$. As $E_s \cap E_q = \emptyset$ for $q \neq s$, it follows $s = q$.

So we have proved

$$i \in A_\mu \leftrightarrow \psi(i) \in E_\mu$$

i.e. that $A$ is r. reducible to $E$. 

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Theorem 3.2. If the creative sequence \( A = A_0, A_1, \ldots \), is reducible to \( B = B_0, B_1, \ldots \), then \( B \) is a creative sequence.

Proof. Let \( A \) be creative under \( \rho \). Therefore, for every disjoint r.e. sequence \( w_{h(\mu)}, \mu = 0, 1, \ldots \), satisfying \( A_{\mu} \cap w_{h(\mu)} = \emptyset \) for all \( \mu \), if \( x \in I(h) \) then

\[
\rho(x) \in \bigcup_{\mu=0}^{\infty} (w_{h(\mu)} \cup A_\mu).
\]

If \( f \) reduces \( A \) to \( B \) then

\[
A_\mu = f^{-1}(B_\mu), \quad \mu = 0, 1, \ldots.
\]

Denote by \( \psi \) the r. function such that for all \( x \in N \)

\[
w_{\psi(x)} = f^{-1}(w_x).
\]

There is a recursive function \( \phi \) such that if \( x \in I(F) \) then \( \phi(x) \in I(\psi(F)) \) (the operation of composition being effective). We shall prove that \( B \) is creative under \( \chi = f(\phi) \).

Let \( w_{k(0)}, w_{k(1)}, \ldots \), be any disjoint r.e. sequence of r.e. sets such that

\[
w_{k(\mu)} \cap B_\mu = \emptyset \quad \text{for all } \mu,
\]

and let \( x \) be an index of the r. function \( k \). We have to prove

\[
\chi(x) \in \bigcup_{\mu=0}^{\infty} (w_{k(\mu)} \cup B_\mu).
\]

By (3.9), using (3.7) and (3.8), we have

\[
A_\mu \cap w_{\psi(k(\mu))} = \emptyset, \quad \text{for all } \mu.
\]

As \( A \) is creative and as \( \phi(x) \in I(\psi(k)) \), we get by (3.6)

\[
\rho(\phi(x)) \in \bigcup_{\mu=0}^{\infty} (A_\mu \cup w_{\psi(k(\mu))}).
\]

From (3.7), (3.8) and (3.12) follows now (3.10).

Corollary 3.2.1. If a sequence \( A \) is weakly creative it is creative.

Proof. Every creative sequence is reducible to \( A \) by Theorem 3.1. By Theorem 3.2, \( A \) is creative.

Theorem 3.3. If a weakly creative sequence \( A = A_0, A_1, \ldots \), is 1-1 reducible to \( B = B_0, B_1, \ldots \), then \( B \) is a weakly creative sequence.

Proof. Let \( A \) be weakly creative under \( \phi \) and let the 1-1 r. function
$f$ reduce $A$ to $B$. There is a recursive $\psi$ such that, for all $x \in \mathbb{N}$,
$$w_{\gamma(\mu, \psi(z))} = f^{-1}(w_{\gamma(\mu, z)}), \quad \mu = 0, 1, \cdots,$$
(Take in (2.3) $Q(\mu, z, x, u) \leftrightarrow x \in f^{-1}(w_{\gamma(\mu, u)}) \wedge z = z.$)

Let $w_{\gamma(0, i)}, w_{\gamma(1, i)}, \cdots$, be a meager sequence. Then $w_{\gamma(0, \psi(i))}, w_{\gamma(1, \psi(i))}, \cdots$, is meager too.

Suppose first that $w_{\gamma(n_0, i)} \neq \emptyset$ and that $w_{\gamma(n_0, i)} \cap B_{n_0} = \emptyset$. Then $w_{\gamma(n_0, \psi(i))} \cap A_{n_0} = \emptyset$ and, as $f$ is 1-1, $w_{\gamma(n_0, \psi(i))}$ is a singleton. Then $\phi(\psi(i)) \in w_{\gamma(n_0, \psi(i))}$ and, as
$$y \in w_{\gamma(n_0, \psi(i))} \leftrightarrow f(y) \in w_{\gamma(n_0, i)},$$
we obtain $f(\phi(\psi(i))) \in w_{\gamma(n_0, i)}$.

If all $w_{\gamma(\mu, i)}$ are empty, from $\phi(\psi(i)) \in \bigcup_{\mu=0}^\infty A_\mu$ we obtain $f(\phi(\psi(i))) \in \bigcup_{\mu=0}^\infty B_\mu$.

This proves that $B$ is weakly creative under $f(\phi(\psi))$.

**Corollary 3.3.1.** Every creative sequence is weakly creative.

**Proof.** By part (3) of Corollary 4 of Cleave's paper [1], every weakly creative sequence is 1-1 r.e. reducible to every creative sequence. By Theorem 3.3 follows the statement.

Corollaries 3.2.1 and 3.3.1 give

**Theorem 3.4.** A sequence is weakly creative iff it is creative.

We point out that using the Definition 3.4 of the paper [2] of Lachlan one can give a definition of $M$-creativity (akin to Lachlan's definition of $M$-coproductivity) which is similar to our definition of weak creativity, but unnecessarily complicated. Namely, starting from the sequence $A = A_0, A_1, \cdots$, Lachlan constructs the sequence $A^* = A_0^*, A_1^*, \cdots$, where

$$A_\mu^* = A_\mu \quad \text{if } A_\mu \text{ is a singleton},$$
$$= \emptyset \quad \text{otherwise}.$$

With this definition, $A$ will be called $M$-creative under $f$ iff $A$ is a r.e. sequence of r.e. sets and iff for all $i$

$$\bigcup_{\mu=0}^\infty (W_{i, \mu} \cap A_\mu^*) = \emptyset \rightarrow \{f(i) \text{ is defined and } W_{i, f(i)} = A_i = \emptyset\}.$$

($W_{i, f(i)}$ is as in §1.) As $M$-creativity is equivalent with creativity it is equivalent with weak creativity.

On the ground of the Theorem 3.4 one can propose the following
definition of creativity, which we shall call $S$-creativity:

A disjoint r.e. sequence $A = A_0, A_1, \ldots$, of r.e. sets is $S$-creative under a recursive $f$ iff for every disjoint sequence $w_0(i), w_1(i), \ldots$, for which $A_\mu \cap w_\gamma(\mu, i) = \emptyset$ for all $\mu$, we have

$$f(i) \in \bigcup_{\mu=0}^{\infty} (A_\mu \cup w_\gamma(\mu, i)).$$

It is not difficult to prove that a sequence is $S$-creative iff it is creative. The implication "$S$-creative $\rightarrow$ creative" is trivial. The converse implication is obtained through a theorem, similar to Theorem 3.3.

References


University of Notre Dame