A LIMIT-POINT CRITERION FOR NONOSCILLATORY
STURM-LIOUVILLE DIFFERENTIAL OPERATORS

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The main point of the present paper is to derive a limit-point criterion from which the criteria of Weyl [6], Friedrichs [3] and Sears [4] follow as special cases. This limit-point criterion is an immediate consequence of a Sturm comparison theorem. It is shown to be equivalent to the criterion of Weyl [6] when use is made of a Liouville-type transformation formula.

Consider the Sturm-Liouville differential operator

\[ L y = -(py')' + qy, \quad X^- < x < X^+, \]

where, in the open interval \((X^-, X^+)\), \(p(x)\) and \(q(x)\) are real and continuous and \(p(x) > 0\).

Let \(x_0\) be any point where \(X^- < x_0 < X^+\).

**Definition (Weyl [6]).** \(L\) is of limit-point type (abbreviated LP) at \(X^+\) if for some constant \(\lambda_0\) not all solutions of \(L y = \lambda_0 y\) are in \(L^2(x_0, X^+)\).

A corresponding definition applies for \(L\) being LP at \(X^-\).

We will freely use the fact, proved by Weyl [6, p. 27], [7, p. 415] (see also [2]), that if all solutions of \(L y = \lambda y\) are in \(L^2\) for some \(\lambda = \lambda_0\), then this is also the case for every \(\lambda\) in the complex plane.

**Theorem.** \(L\) defined by

\[ L y = -(py')' + qy \]

is LP at \(X^+\) if there is a comparison operator

\[ \hat{L} v = -(\hat{p}v')' + \hat{q}v \]

such that (i) \(\hat{L}\) is LP and nonoscillatory at \(X^+\) and

(ii) \(q - \hat{q}\) is bounded below at \(X^+\).

**Proof.** By hypothesis (i) there is a solution, \(v\), of \(\hat{L} v = 0\) which is strictly positive and not in \(L^2\) on some interval \((x_0, X^+)\).

By hypothesis (ii) there is a constant \(\lambda_0\) such that for all \(x \geq x_0\),

\[ q - \hat{q} \geq \lambda_0. \]

Presented to the Society, September 1, 1966; received by the editors June 24, 1966.

1 This research was completed under U. S. Navy Contract No. Nonr 3360 (01).
Define $y$ to be the solution of $Ly = \lambda_0 y$ which satisfies the initial conditions
\[ y(x_0) = v(x_0), \quad y'(x_0) = v'(x_0). \]
Then by a Sturm comparison theorem (see, for example, [1] or [5])
\[ y(x) \geq v(x) \quad \text{for } x \geq x_0. \]
Hence $y(x)$ is not in $L^2(x_0, X^+)$. Q.E.D.

It is easily seen that the above theorem is also valid if one replaces $X^+$, wherever it occurs, by $X^-$. To verify this it is sufficient to simply apply the above theorem to the problem obtained after the change of variables $s = -x$.

We now show that the LP criteria of Weyl [6], Friedrichs [3] and Sears [4] follow from the above by proper selection of the comparison operator.

For the first two LP criteria let $\hat{q} = 0$. The solutions of $\hat{L}v = 0$ are then linear combinations of $v_1 = 1$ and $v_2 = \int_{x_0}^x (\phi(s))^{-1} ds$. Clearly $\hat{L}$ is nonoscillatory. Furthermore, $\hat{L}$ will be LP at $X^+$ if either $X^+ = \infty$ (since then $v_1 = 1$ is not in $L^2$) or if $X^+ < \infty$ and $v_2$ is not in $L^2$. Thus we have rederived the following:

**Corollary 1 (Weyl [7, p. 27]).** $L$ is LP at $X^+ = \infty$ if $q$ is bounded below at $X^+ = \infty$.

**Corollary 2 (Friedrichs [3, p. 13]).** $L$ is LP at $X^+ < \infty$ if $q$ is bounded below at $X^+$ and $\int_{x_0}^x (\phi(s))^{-1} ds$ is not in $L^2(x_0, X^+)$. 

For the Sears LP criterion let $\hat{p} = 1$ and $\hat{q} = c/x^2$ where $c$ is a constant as yet unspecified. The solutions of $\hat{L}v = 0$ are then linear combinations of $v_1 = x^{\gamma_1}$ and $v_2 = x^{\gamma_2}$ where $\gamma_{1,2} = 1/2 \pm (c+1/4)^{1/2}$. Clearly $\hat{L}$ is nonoscillatory if $c > -1/4$. If $c \geq 3/4$ then $v_2$ is not in $L^2(0, x_0)$ so that $\hat{L}$ is then LP at $X^- = 0$. We have thus rederived

**Corollary 3 (Sears [4]).** $L$ defined by
\[ Ly = -y'' + qy \]
is LP at $X^- = 0$ if $q - 3/4x^2$ is bounded below at $X^- = 0$.

It is interesting to note that the above theorem can be derived from the LP criterion of Weyl in Corollary 1 when use is made of the following Liouville-type transformation:

**Lemma.** (See, for example, [3, p. 16].)
\[ Ly = - (py')' + qy, \quad x_0 < x < X^+, \]
becomes
\[ Lu = \frac{d}{dt}(p(du/dt)) + qu, \quad 0 < t < T^+, \]

by the transformation
\[ y = vu, \quad t(x) = \int_{x_0}^x v^2(s)ds \]

with \( v \) as yet unspecified where
\[ \tilde{p} = v^4p, \quad \tilde{q} = q - \frac{(pv')'}{v}, \quad T^+ = \int_{x_0}^{X^+} v^2(s)ds. \]

Furthermore,
\[ \int_{x_0}^{X^+} y^2(x)dx = \int_{0}^{T^+} u^2(t)dt. \]

Thus \( \tilde{L} \) is LP at \( T^+ \) if and only if \( L \) is LP at \( X^+ \).

Now let \( v \) in the above Lemma be a solution of the comparison equation
\[ \tilde{L}v = - (pv')' + \tilde{q}v = 0. \]

Then clearly \( \tilde{q} = q - \tilde{q} \).

If \( \tilde{L} \) is LP at \( X^+ \) then one can select \( v \) so that \( T^+ = \infty \).

If \( \tilde{L} \) is nonoscillatory at \( X^+ \) then \( \tilde{p} = v^4p > 0 \).

Thus the hypothesis of the theorem insures that one can apply the LP criterion of Weyl to conclude that \( L \) is LP at \( T^+ = \infty \). Hence by the above Lemma \( L \) is LP at \( X^+ \). Q.E.D.

One can generalize Corollaries 2 and 3 by introducing a new independent variable

\[ t(x) = \int_{x_0}^x ds/p(s) \]

and setting
\[ \tilde{q}(x) = h(t)/p(x), \]

where the function \( h(t) \) is as yet unspecified. Then
\[ \tilde{p}(x)\tilde{L}v = - d^2v/dt^2 + h(t)v. \]

Each \( h(t) \) for which \( \tilde{L} \) is nonoscillatory and LP at \( X^+ \) or \( X^- \) then leads, via the above theorem, to an LP criterion for \( L \). For example, if one selects
\[ h(t) = C/t^2 \]
then $\hat{L}v=0$ has solutions $t^{\gamma_1}$ and $t^{\gamma_2}$ where $\gamma_1$ and $\gamma_2$ are the values indicated in the above discussion of Corollary 3. One thus concludes the following:

**Corollary 4.** $L$ defined in (1) is LP at $X^+$ ($X^-$) if $C \geq 1/4$ is such that (i) at least one of $t^{\gamma_1}$ and $t^{\gamma_2}$ is not in $L^2(x_0, X^+)$ ($L^2(x_0, X^-)$) and (ii) $q(x) - C/p(x)t^2$ is bounded below at $X^+$ ($X^-$) where $t$ is defined in (2).

Corollary 2 corresponds to the selection $C = 0$. Then $\gamma_1 = 1$ and $\gamma_2 = 0$ so that condition (i) becomes simply that $t$ is not in $L^2$.

Corollary 3 corresponds to the selection $p(x) = 1$. This results in $t(x) = x$ and the optimum value of $C$ is seen to be $3/4$.

As a further example of Corollary 4 let $p(x) = x^\alpha$ where $\alpha$ is a real constant. (Corollary 2 will not be applicable to this case if $\alpha < 3/2$ since then $t(x)$ will be in $L^2$.) From (2), (3) and (5)

$$q(x) = \frac{K}{x^{2-\alpha}}$$

where $K = (1-\alpha)^2C$. It is then easily seen that $\hat{L}$ is nonoscillatory and LP at $X^-=0$ if $K \geq (3-2\alpha)/4$. This then proves

**Corollary 5.** $L$ defined by (1) is LP at $X^- = 0$ if $p = x^\alpha$, $\alpha < 3/2$ and $q - (3-2\alpha)/4x^{2-\alpha}$ is bounded below at $X^- = 0$.

Corollary 3 corresponds to the special case $\alpha = 0$.

In order to facilitate comparison with Corollary 2 one can rewrite Corollary 5 by replacing $x$ by $X^+ - x$ and $X^- = 0$ by $X^+ < \infty$.

A selection of $h(t)$ in (3) more general than (5), and yet which is readily dealt with, is one for which $\hat{L}v=0$ has a regular singularity at $T$ where $T = \lim t(x)$ as $x \to X^-$. Thus if $T = 0$ let

$$h(t) = t^{-2}(C + C_1 t + \cdots + C_k t^k)$$

while if $T = \infty$ let

$$h(t) = t^{-2}(C + C_1 t^{-1} + \cdots + C_k t^{-k}).$$

In either case, as long as $\gamma_1 - \gamma_2$ is not an integer, $t^{\gamma_1}$ and $t^{\gamma_2}$ are now the leading terms of two solutions of $\hat{L}v=0$. Hence in Corollary 4 one can replace (ii) by (ii)' $q(x) - h(t)/p(x)$ is bounded below at $X^+$ ($X^-$) where $C$ is described in Corollary 4 but $C_1, C_2, \cdots, C_k$ are unrestricted.

**References**


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