GROWTH ESTIMATES OF CONVEX FUNCTIONS
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In this paper we generalize a result of Loewner's by use of an iterated form of the Schwarz Lemma. This tool is used repeatedly to obtain several results on the growth of convex analytic functions.

**Preliminaries.** By a convex function we will mean a univalent analytic function, defined on the unit disc, whose range is a convex point set. Such a function \( f \) satisfies the differential inequality:

\[
\text{Re}\left\{ \frac{zf''}{f'} \right\} > -1 \quad \text{for all } z \text{ in the unit disc.}
\]

By a starlike function we will mean a univalent analytic function, defined on the unit disc, whose range is a starlike set with respect to the image of the origin. That is, if \( \xi \) is the image of 0, and \( w \) lies within the range, then the line segment joining \( \xi \) and \( w \) lies within the range as well. If we assume that \( f(0) = 0 \), such a function satisfies the differential inequality:

\[
\text{Re}\left\{ \frac{zf'}{f} \right\} > 0 \quad \text{for all } z \text{ in the unit disc.}
\]

A trivial consequence of the above is that a function \( f \) is convex if and only if \( zf' \) is starlike. As a result, all theorems on convex functions will yield results on starlike functions, and conversely.

In what follows, we shall assume that the functions under consideration are normalized. That is, \( f(0) = 0, f'(0) = 1 \). This is no essential restriction, for if \( g(z) \) is any univalent function defined on the unit disc, then

\[
f(z) = \frac{(g(z) - g(0))}{g'(0)}
\]

is a normalized, univalent function. We shall for simplicity occasionally assume in addition that \( f''(0) \geq 0 \). Again, this is no restriction, for by appropriate choice of \( \theta \), \( f(z) = e^{-i\theta} g(e^{i\theta} z) \) will have a real and nonnegative second derivative at \( 0 \).

Loewner [2] proved that for a convex function \( f(z) = z + \cdots, |f'(z)| \leq 1/(1-r)^2, |z| = r \). Theorem 1 improves upon this result to give growth estimates of \( f' \) involving the second coefficient in the power series expansion of \( f \). First we state and prove an iterated form of Schwarz' Lemma. The method of proof goes back at least as far as Landau [1, p. 307]. The lemma itself appeared in [3].

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Lemma 1. Let \( f(z) = a_1 z + \cdots \) be an analytic map of the unit disc into itself. Then \( |a_1| \leq 1 \), and
\[
|f(z)| \leq r(r + |a_1|)/(1 + |a_1|r).
\]

Equality holds at some \( z \neq 0 \) if and only if
\[
f(z) = \frac{e^{-i\theta} z + a_1 e^{i\theta}}{1 + \bar{a}_1 e^{-i\theta}}, \quad t \geq 0.
\]

Proof. That \( |a_1| \leq 1 \) is an immediate consequence of Schwarz’ Lemma. We mention it so that the inequality here becomes an improvement on Schwarz’ Lemma, in that the estimate is better when \( |a_1| < 1 \).

Let \( g(z) = f(z)/z = a_1 + a_2 z + \cdots \). Then the maximum principle shows that \( |g(z)| < 1 \) in the unit disc.

Let \( h(z) = (g(z) - a_1)/(1 - \bar{a}_1 g(z)) = b_1 z + \cdots \). Then as \( |g| < 1 \), also \( |h| < 1 \). Further, \( h(0) = 0 \), and thus, by the familiar form of Schwarz’ Lemma,
\[
|h(z)| \leq r.
\]
Equality holds for some \( z \neq 0 \) if and only if \( h(z) = e^{i\theta}z \). But
\[
|f(z)| = \left| \frac{h(z) + a_1}{1 + \bar{a}_1 h(z)} \right| \leq r \left| \frac{h(z)}{1 + |a_1 h(z)|} \right| \leq r \left( \frac{r + |a_1|}{1 + |a_1|} \right).
\]

Further, it is clear that equality can hold at a point \( z \) only if \( |h(z)| = |z| \), from which it follows that
\[
f(z) = \frac{e^{i\theta} z + a_1 e^{i\theta}}{1 + \bar{a}_1 z e^{i\theta}}.
\]

Theorem 1. Let \( f(z) = z + a_2 z^2 + \cdots \) be convex in the unit disc. Then
\[
|f'(z)| \leq \left( \frac{1}{1 - r^2} \right) \left[ \frac{1 + r}{1 - r} \right]^p,
\]
where \( |a_2| = \rho \). Equality holds at some \( z \neq 0 \) if and only if
\[
f(z) = \frac{e^{-i\theta}}{2\rho} \left( \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} \right)^p - 1.
\]

This function (2) maps the unit disc onto a sector of opening angle \( p\pi \), the apex of the sector lying at a point of modulus \( 1/2\rho \), and 0 lying on the axis of symmetry.

Proof. That (2) maps the unit disc as indicated above can be seen
through the Schwarz-Christoffel representation for mappings onto polygons.\footnote{See, for example, Hille, \textit{Analytic function theory}. II, Ginn, Boston, Mass., p. 372.}

\[ f'(z) = (1 + e^{i\alpha}z)^{\rho-1}(1 - e^{i\alpha}z)^{-\rho-1}. \]

Consequently, \( f \) maps onto a 2-sided polygon (sector) whose exterior angle at the apex is \((1 - \rho)\pi\). Hence the interior angle is \( \rho\pi \). The remaining details of this mapping are easily verified.

We can, as before, perform a rotation in the \( z \)- and \( w \)-planes so that \( a_2 = \rho > 0 \). So \( f(z) = z + \rho z^2 + a_3 z^3 + \cdots \). Let \( \psi(r) = (1 - r)^{1+\rho}(1 + r)^{1-\rho}. \)

Let

\[ \phi(z) = \log |f'(z)| + \log \psi(r). \]

We shall show that \( \phi(z) \leq 0 \). In fact, we shall show even more, that \( \partial \phi/\partial r \leq 0 \) for \( z \neq 0 \).

Let us note that \( \phi(0) = 0 \), and that \( \phi \) is continuous in \( z \). Now

\[ \frac{\partial \phi}{\partial r} = (\partial/\partial r) \log |f'(z)| - (1 + \rho)/(1 - r) + (1 - \rho)/(1 + r). \]

But

\[ r (\partial/\partial r) \log |f'(z)| = \text{Re} \{zf''/f'\}, \]

so

\[ r (\partial \phi/\partial r) = \text{Re} \{zf''/f'\} - 2r(\rho + \rho)/(1 - r^2). \]

Now, if \( h(z) = zf''/f' \), we have \( 1 + h(z) \) is a function of positive real part, or \( 1 + h = (1 + g)/(1 - g) \), for \( |g| < 1 \), \( g(z) = \rho z + \cdots \),

\[ h = 2g/(1 - g). \]

By Lemma 1, \( |g(z)| \leq r(\rho + \rho)/(1 + r\rho) \), and thus

\[ |h(z)| \leq 2r(\rho + \rho)/(1 - r^2). \]

Then

\[ r (\partial \phi/\partial r) = \text{Re} \{zf''/f'\} - 2r(\rho + \rho)/(1 - r^2), \]

and

\[ \text{Re} \{zf''/f'\} \leq |zf''/f'| = |h| \leq 2r(\rho + \rho)/(1 - r^2). \]

Thus by (3)

\[ r (\partial \phi/\partial r) \leq 0. \]

This completes the proof that
\[ |f'(z)| \leq \frac{1}{1 - r^2} \left[ \frac{1 + r}{1 - r} \right]^p \]

If equality is attained at a point, an investigation of the above proof shows that \( h(z) \) must achieve the estimate for it along a ray. Then, as in Lemma 1, \( h \) is determined up to a rotational constant:

\[ h(z) = 2e^{it}(z + \rho)/(1 - z^2). \]

From this it follows that

\[ f(z) = (e^{-it}/2\rho) \left( \left[ (1 + e^{it})/(1 - e^{it}) \right]^p - 1 \right). \]

We observe the following well-known result:

**Corollary 1.1.** Let \( f(z) = z + az^3 + az^5 + \cdots \) be an odd convex function. Then

\[ |f'(z)| \leq 1/(1 - r^2). \]

**Proof.** An odd function has vanishing even coefficients, and on setting \( a_2 = p = 0 \) in (1) we obtain the result.

**Corollary 1.2.** Let \( f(z) = z + \rho z^2 + az^3 + \cdots \) be convex in the unit disc. Then

\[ |f(z)| \leq (1/2\rho) \left( \left[ (1 + r)/(1 - r) \right]^p - 1 \right). \]

This estimate is sharp.

**Proof.** As we remarked in the preliminaries, it is no loss of generality to assume as we have here that \( a_2 = \rho \geq 0 \). The result follows on integration of (1). The same function (4) is extremal for both Theorem 1 and this corollary.

**Corollary 1.3.** Let \( f(z) = z + 2p z^2 + az^3 + \cdots \) be starlike in the unit disc, \( 0 \leq p \leq 1 \). Then

\[ |f(z)| \leq \frac{r}{1 - r^2} \left[ \frac{1 + r}{1 - r} \right]^p. \]

**Proof.** We observed earlier that \( g \) is convex if and only if \( zg'(z) \) is starlike. Or, what is the same, \( f \) is starlike if and only if \( \int_0^1 (f(z)/z)dz \) is convex. Hence

\[ \int_0^1 (f(z)/z)dz = z + \rho z^2 + \frac{a_3}{3} z^3 + \cdots \]

is convex, and, by Theorem 1,

*See, for example, Nehari, *Conformal mapping*, McGraw-Hill, New York, p. 238.*
\[ \left| \frac{f(z)}{z} \right| \leq \frac{1}{1 - r^2} \left[ \frac{1 + r^2}{1 - r} \right]^\rho. \]

The result then follows. The function
\[ \left( \frac{z}{1 - z^2} \right)^\rho \]
is starlike and achieves this estimate. To determine the range of this function, we again use the Schwarz-Christoffel representation. Computing the derivative of this function,
\[ f'(z) = \frac{1 + 2\rho z + z^2}{(1 - z)^{2+\rho}(1 + z)^{2-\rho}} \]
\[ = \frac{(1 + e^{i\theta}z)(1 + e^{-i\theta}z)}{(1 - z)^{2+\rho}(1 + z)^{2-\rho}}, \quad e^{i\theta} + e^{-i\theta} = 2\rho. \]

The range of \( f \) is thus a 4-sided polygon with exterior angles \((2+\rho)\pi, -\pi, (2-\rho)\pi, -\pi\). It is then easily seen that this polygon is the entire plane minus two radial slits extending to \( \infty \). The two slits make an angle of \((1-\rho)\pi\) with one another.

**Theorem 2.** Let \( f(z) = z + \rho z^2 + az^3 + \cdots \) be convex in the unit disc, \( \rho \geq 0 \). Then
\[ |f'(z)| \geq \frac{1}{1 + 2\rho r + r^2}. \]

**Proof.**
\[ r(\partial/\partial r) \log |f'(z)| = \text{Re} \left\{ \frac{zf''/f'}{f'} \right\}. \]
Now \( 1 + zf''/f' = (1+g)/(1-g) \), for some \( |g(z)| < 1 \), and
\[ 1 + \text{Re} \left\{ \frac{zf''}{f'} \right\} = \text{Re} \left\{ \frac{1 + g}{1 - g} \right\} = \frac{1 - g\bar{g}}{|1 - g|^2} \]
\[ \geq \frac{1 - g\bar{g}}{(1 + |g|)^2} = \frac{1 - |g|}{1 + |g|}. \]

Hence
\[ \text{Re} \left\{ \frac{zf''/f'}{f'} \right\} \geq (-2|g|)/(1 + |g|). \]

Applying Lemma 1 to \( g \),
\[ |g(z)| \leq r(r + \rho)/(1 + \rho r). \]

Hence from (6) we have
\[ \text{Re} \left\{ \frac{zf''/f'}{f'} \right\} \geq \frac{-2r(r + \rho)/(1 + \rho r)}{1 + r(r + \rho)/(1 + \rho r)} = \frac{-2r(r + \rho)}{1 + 2\rho r + r^2}. \]
But then by (5),
\[(\partial/\partial r) \log |f'(z)| \geq -2(r + \rho)/(1 + 2\rho r + r^2).\]

On integrating (7) with respect to \(r\), and then exponentiating, we obtain
\[|f'(z)| \geq 1/(1 + 2\rho r + r^2).\]

If \(\rho < 1\), the function
\[f(z) = \frac{1}{2i \sin \theta} \log \left( \frac{1 - e^{-i\theta}z}{1 - e^{i\theta}z} \right), \quad \rho = \cos \theta\]
is convex and achieves the estimate of the theorem along the negative real axis from 0 to \(-1\). By considering the Schwarz-Christoffel representation, it is seen that this function maps the unit disc onto an infinite strip, of width \(\pi/(2 \sin \theta)\), with the image of the origin lying at distances from the two parallel boundary lines \((\pi - \theta)/(2 \sin \theta), \theta/(2 \sin \theta)\) respectively.

If \(\rho = 1\), the function \(f(z) = z/(1 - z)\) achieves the estimate of the theorem along the negative real axis from 0 to \(-1\). This function maps the unit disc onto a half plane.

**Corollary 2.1.** Let \(f(z) = z + \rho z^2 + a_3 z^3 + \cdots\) be convex in the unit disc. Then
\[|f(z)| \geq \frac{1}{(1 - \rho^2)^{1/2}} \tan^{-1} \left( \frac{r + \rho}{(1 - \rho^2)^{1/2}} \right), \quad \rho \geq 0.\]

**Proof.** Let \(\Gamma\) be the straight line segment joining 0 and \(f(z)\). Since \(f\) is convex, \(\Gamma\) lies in the range of \(f\). Let \(\gamma = f^{-1}(\Gamma) = \{z : f(z) \in \Gamma\}\). Then \(\gamma\) is an arc in the unit disc joining 0 and \(z\). It follows that
\[|f(z)| = \int_{\gamma} |f'(z)| \, dz \geq \int_0^r |f'(s)| \, ds \geq \int_0^r \frac{ds}{1 + 2\rho s + s^2}.\]
The stated result then follows.

Now consider the function
\[f(z) = \frac{1}{2i \sin \theta} \log \left( \frac{1 - e^{-i\theta}z}{1 - e^{i\theta}z} \right) = \int_0^s \frac{d\xi}{(1 - e^{i\theta} \xi)(1 - e^{-i\theta} \xi)},\]
which was discussed in the preceding theorem. For \(e^{i\theta} + e^{-i\theta} = 2\rho\), it is clear that \(f'(r) = (1 + 2\rho r + r^2)^{-1}\), which is the lower estimate for \(|f'(z)|\). Hence in our proof above, if \(\Gamma\) is the straight line segment joining 0 and \(f(r)\), then \(\gamma\) lies in the positive real axis. Hence all the
inequalities can be replaced by equalities for this function, which shows the estimate to be sharp.

**Corollary 2.2.** Let \( f(z) = z + \rho z^2 + a_3 z^3 + \cdots \) be starlike in the unit disc. Then

\[
|f(z)| \leq r/(1 + \rho r + r^2).
\]

**Proof.** The proof is exactly the same as the proof of Corollary 1.3. The function which shows the estimate to be sharp is \( z/(1 + \rho z + z^2) \).

**Theorem 3.** Let \( f(z) = z + \rho z^2 + a_3 z^3 + \cdots \) be convex in the unit disc. Then

\[
|f''(z)| \leq \frac{2(r + \rho)}{(1 - r^2)^2} \left[ \frac{1 + r}{1 - r} \right]^p.
\]

This result is sharp.

**Proof.** As in the proof of Theorem 2, there exists a function \( g \), \(|g| < 1\), \( g(z) = \rho z + \cdots \), so that

\[
\frac{zf''}{f'} = \frac{2g}{1 - g}.
\]

\[
\left| \frac{zf''}{f'} \right| = \left| \frac{2g}{1 - g} \right| \leq \frac{2|g|}{1 - |g|} \leq \frac{2r(r + \rho)}{1 - r^2},
\]

on applying Lemma 1 to \( g \). Hence

\[
|f''| \leq \frac{2r(r + \rho)}{1 - r^2} \left| \frac{f'}{r} \right| \leq \frac{2(r + \rho)}{(1 - r^2)^2} \left[ \frac{1 + r}{1 - r} \right]^p
\]

when we apply the estimate of Theorem 1 for \(|f'|\). Again, the extremal function of Theorem 1 achieves this estimate.

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**References**


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