GROTHENDIECK GROUPS AND DIVISOR GROUPS

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0. Introduction. Before stating the results in this note, it is necessary to introduce some notation. \( A \) is a noetherian integral domain which is integrally closed in its quotient field \( K \). \( \Sigma \) is a central simple finite-dimensional \( K \)-algebra, \( D \) is a central division \( K \)-algebra and \( V \) is a finitely generated right \( D \) vector space such that \( \Sigma = \text{Hom}_D(V, V) \) (so also \( D = \text{Hom}_\Sigma(V, V) \)).

Let \( \Lambda \) be an \( A \)-order in \( \Sigma \). \( \mathfrak{M}(\Lambda) \) denotes the category of left finitely generated \( \Lambda \)-modules, \( \mathfrak{I}(\Lambda) \) the Serre subcategory of \( \mathfrak{M}(\Lambda) \) consisting of \( \Lambda \)-torsion left \( \Lambda \)-modules. \( \mathfrak{P}(\Lambda) \) is the Serre subcategory of \( \mathfrak{I}(\Lambda) \) consisting of the pseudo-nul left \( \Lambda \)-modules, where a pseudo-nul module \( M \) is one for which \( M_p = A_p \otimes A M = 0 \) for all prime ideals \( p \) of \( A \) of height at most one. The category \( \mathfrak{M}/\mathfrak{P}(\Lambda) \) is formed by taking as objects the objects of \( \mathfrak{M}(\Lambda) \) and for \( M, N \) in \( \mathfrak{M}(\Lambda) \), defining \( \text{Hom}_{\mathfrak{M}/\mathfrak{P}}(M, N) \) to be the direct limit of \( \text{Hom}_{\mathfrak{M}}(M', N') \) taken over those \( M' \) and \( N' \) such that \( M/M' \) is in \( \mathfrak{P} \) and \( N' = N/N'' \) with \( N'' \) in \( \mathfrak{P} \). \( \mathfrak{I}/\mathfrak{P}(\Lambda) \) is formed in a similar fashion. The first result may now be stated as follows:

**Theorem 1.** Let \( A, \Sigma, D \) be as above. Let \( \Lambda_1 \) and \( \Lambda_2 \) be maximal orders in \( \Sigma \), and \( \Gamma \) a maximal order in \( D \). Then there are functors

\[
F(\Lambda_1, \Lambda_2): \mathfrak{M}(\Lambda_1) \to \mathfrak{M}(\Lambda_2),
\]

\[
G(\Lambda_2, \Gamma): \mathfrak{M}(\Lambda_2) \to \mathfrak{M}(\Gamma),
\]

which induce isomorphisms of the categories

\[
\mathfrak{M}/\mathfrak{P}(\Lambda_1) \to \mathfrak{M}/\mathfrak{P}(\Lambda_2) \to \mathfrak{M}/\mathfrak{P}(\Gamma),
\]

\[
\mathfrak{I}/\mathfrak{P}(\Lambda_1) \to \mathfrak{I}/\mathfrak{P}(\Lambda_2) \to \mathfrak{I}/\mathfrak{P}(\Gamma).
\]

If \( \mathcal{C} \) is an abelian category, \( K^0(\mathcal{C}) \) denotes the Grothendieck group of \( \mathcal{C} \). It can be defined as follows: For each \( C \) in \( \mathcal{C} \) there is an \( f(C) \) in \( K^0(\mathcal{C}) \), an abelian group, such that if \( 0 \to C' \to C \to C'' \to 0 \) is an exact sequence in \( \mathcal{C} \), then \( f(C) = f(C') + f(C'') \). Furthermore, if \( G \) is any abelian group and for each \( C \) in \( \mathcal{C} \) there is a \( g(C) \) in \( G \) such that \( g(C) = g(C') + g(C'') \) on exact sequences in \( \mathcal{C} \) then there is a unique homomorphism \( h: K^0(\mathcal{C}) \to G \) such that \( g = hf \). Let \( G_t(\Lambda) = K^0(\mathfrak{I}/\mathfrak{P}(\Lambda)) \) and \( G(\Lambda) = K^0(\mathfrak{M}/\mathfrak{P}(\Lambda)) \). An immediate corollary

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to Theorem 1 is

**Corollary.** The functors $F$ and $G$ induce isomorphisms

$$G_t(\Lambda_1) \to G_t(\Lambda_2) \to G_t(\Gamma),$$

$$G(\Lambda_1) \to G(\Lambda_2) \to G(\Gamma).$$

In case $A$ is a Dedekind domain these results are known, so in a sense Theorem 1 may be considered to be a generalization of the Morita Theorems which give these isomorphisms in this case (see [5]).

If $M$ is an $A$-lattice in $\Sigma$, define $M^{-1} = \{ x \in \Sigma : MxM \subseteq M \}$. Let $\Delta$ be a maximal order in $\Sigma$. Let $I(\Delta)$ denote the set of $A$-lattices in $\Sigma$ which are both left and right $A$-modules. Goldman in [6] defined $D(\Delta)$, the group of divisors of $\Lambda$, to be the abelian group obtained from $I(\Delta)$ by the equivalence relation (quasi-equality for two-sided fractionary $A$-ideals).

"$M \sim N$ in $I(\Lambda)$ iff $M^{-1} = N^{-1}$." Thus $D(\Delta) = I(\Delta)/\sim$, with multiplication given by $(M, N) \mapsto \overline{MN}$. Goldman proves that $D(\Lambda_1)$ is naturally isomorphic to $D(\Lambda_2)$ when $\Lambda_1$ and $\Lambda_2$ are maximal orders in $\Sigma$. The second result of this note is

**Theorem 2.** $D(\Lambda)$ is isomorphic to $G_t(\Lambda)$.

Theorem 2 and the corollary to Theorem 1 yield the important, but not surprising, result, namely the

**Corollary.** If $\Lambda$ is a maximal order in $\Sigma$, and $\Gamma$ a maximal order in $D$, then $D(\Lambda)$ is (naturally) isomorphic to $D(\Gamma)$.

Thus considerations of $D(\Lambda)$ are reduced to considerations of $D(\Gamma)$, but $\Gamma$ is in a division algebra.

1. **Proof of Theorem 1.** The notations of §0 are retained here. Let $\Lambda$ and $\Omega$ be maximal $A$-orders in $\Sigma$. The conductor, $\{ x \in \Sigma : \Omega x \subseteq \Lambda \}$, is denoted by $\Lambda : \Omega$. It is an $A$-lattice in $\Sigma$ which is a right ideal in $\Lambda$ and a left $\Omega$-module. Define $F(\Lambda, \Omega)$, $F(\Lambda, \Omega)(M) = \Lambda : M \otimes_A M$ for the left $\Lambda$-module $M$. Certainly $F(\Lambda, \Omega)$ is a functor. Since $A_p$ is a flat $A$-module for each prime ideal $p$ of $A$, it is clear that $A_p \otimes_A F(\Lambda, \Omega) = F(\Lambda_p, \Omega_p)$ for each prime ideal $p$ of $A$. Hence $F$ takes torsion modules to torsion modules, and pseudo-nul modules to pseudo-nul modules, and consequently induces functors

$$F'(\Lambda, \Omega) : \mathfrak{M}/\mathfrak{p}(\Lambda) \to \mathfrak{M}/\mathfrak{p}(\Omega),$$

$$F''(\Lambda, \Omega) : \mathfrak{J}/\mathfrak{p}(\Lambda) \to \mathfrak{J}/\mathfrak{p}(\Omega).$$

($F''$ is induced by $F'$.)
To show that $F'$ (and hence $F''$) is an isomorphism, it is sufficient to construct a functorial inverse. But, consider the natural transformation

$$F(\Omega, \Lambda)F(\Lambda, \Omega) \rightarrow I_{\mathfrak{M}(\Lambda)}$$

given by $(\Omega: \Lambda) \otimes_{\Omega} (\Omega: \Lambda) \otimes_{\Lambda} M \rightarrow M: \omega \otimes \lambda \otimes m \rightarrow \omega \lambda m$. Upon localizing at a height one or less prime ideal of $\Lambda$, one obtains an identification; that is, $F(\Omega_p, \Lambda_p)F(\Lambda_p, \Omega_p) = I$. For in case $p=0$, $\Omega_p = \Sigma = \Lambda_p$, and in the other cases, $\Lambda_p$ is a discrete rank-one valuation ring, so $\Lambda_p: \Omega_p = u\Lambda_p = \Omega_pu$ and $\Omega_p: \Lambda_p = u^{-1}\Omega_p = \Lambda_pu^{-1}$, where $u$ is a unit in $\Sigma$ (by 3.4 of [1]). Hence $F'$ (and so $F''$) is an isomorphism.

Using the same arguments, one shows that $F'(\Lambda, \Omega)F'(\Omega, \Omega') = F'(\Lambda, \Omega')$ for maximal $A$-orders in $\Sigma$. This says that the isomorphisms are natural.

Before proving the second part of Theorem 1, a generalization of Proposition 4.2 of [1] is needed.

The proof is exactly as in [1]. Proposition 4.1 of [1] and its proof remain valid when Hom is replaced by Hom$_\Gamma$ and $\otimes$ by $\otimes_\Gamma$, so it can be used as in the proof of [1, Proposition 4.2].

**Proposition 1.** Let $A$ be a noetherian integrally closed integral domain with quotient field $K$. Let $\Sigma$ be a finite-dimensional central simple $K$-algebra. Suppose $\Sigma = \text{Hom}_D(V, V)$ where $D$ is a central division $K$-algebra and $V$ a finite-dimensional right $D$-module. An $A$-order $\Lambda$ in $\Sigma$ is maximal if, and only if, there is a maximal $A$-order $\Gamma$ in $D$ and a right $\Gamma$-submodule $E$ of $V$ which is a reflexive $A$-lattice such that $\Lambda = \text{Hom}_\Gamma(E, E)$. In this case $\Gamma = \text{Hom}_\Lambda(E, E)$.

Let $\Lambda$ be a maximal order in $\Sigma$ and let $E$ and $\Gamma$ be as in Proposition 1. Define $G(\Lambda, \Gamma): \mathfrak{M}(\Lambda) \rightarrow \mathfrak{M}(\Gamma)$ by $G(\Lambda, \Gamma)(M) = \text{Hom}_\Gamma(E, \Gamma) \otimes_{\Lambda} M$. The localization arguments used above show that $G(\Lambda, \Gamma)$ preserves torsion and pseudo-nullity, so $G$ induces

$$G'(\Lambda, \Gamma): \mathfrak{M}/\mathfrak{F}(\Lambda) \rightarrow \mathfrak{M}/\mathfrak{F}(\Gamma),$$

$$G''(\Lambda, \Gamma): \mathfrak{F}/\mathfrak{F}(\Lambda) \rightarrow \mathfrak{F}/\mathfrak{F}(\Gamma).$$

There is also the functor $G(\Gamma, \Lambda): \mathfrak{M}(\Gamma) \rightarrow \mathfrak{M}(\Lambda)$ defined by $G(\Gamma, \Lambda)(N) = E \otimes_\Gamma N$. As before, there are natural transformations

$$G(\Lambda, \Gamma)G(\Gamma, \Lambda) \rightarrow I_{\mathfrak{M}(\Lambda)},$$

$$G(\Gamma, \Lambda)G(\Lambda, \Gamma) \rightarrow I_{\mathfrak{M}(\Lambda)}.$$
tions so
\[ G'(\Lambda, \Gamma)G'(\Gamma, \Lambda) = I_{\mathfrak{M}/\mathfrak{P}(\Gamma)}; \]
\[ G'(\Gamma, \Lambda)G'(\Lambda, \Gamma) = I_{\mathfrak{M}/\mathfrak{P}(\Lambda)}. \]

This concludes the proof of Theorem 1.

Heller and Reiner in [4], [5] discuss the exact sequences
\[ K^1(\Sigma) \rightarrow G_t(\Lambda) \rightarrow G(\Lambda) \rightarrow K^0(\Sigma) \rightarrow 0, \]
\[ K^1(D) \rightarrow G_t(\Gamma) \rightarrow G(\Gamma) \rightarrow K^0(D) \rightarrow 0, \]

where \( A \) is a Dedekind domain.

The corollary to Theorem 1 generalizes the discussion on pp. 351-352 of [5], i.e. it implies that these are isomorphic sequences for any noetherian integrally closed integral domain \( A \).

Another application of the corollary to Theorem 1 is

**Proposition 2.** Let \( A \) be a noetherian integrally closed integral domain with quotient field \( K \). Let \( V \) be a finite-dimensional vector space over \( K \) and let \( \Sigma = \text{Hom}_K(V, V) \). Let \( \Lambda \) be a maximal order in \( \Sigma \). Then
\[ G_t(\Lambda) = D(\Lambda) \quad \text{(divisor group of } A), \]
\[ G(\Lambda) = C(\Lambda) \oplus \mathbb{Z} \quad (C(\Lambda) = \text{class group of } A). \]

**Proof.** By the corollary to Theorem 1, \( G_t(\Lambda) = G_t(\Lambda) \) and \( G(\Lambda) = G(\Lambda) \). By Proposition 11 of [3, §4, no5], \( G_t(\Lambda) = D(\Lambda) \). By Proposition 17 of [3, §4, no8], \( G(\Lambda) = C(\Lambda) \oplus \mathbb{Z} \).

**Remark.** Theorem 2 is a generalization of this proposition.

2. **Proof of Theorem 2.** The proof of the theorem is exactly the proof of Proposition 11 of [3, §4, no5] modified to the present situation.

Let \( \Lambda \) be a maximal \( A \)-order in \( \Sigma \). For each prime (two-sided) ideal \( \mathfrak{P} \) of \( \Lambda \) of height one let div \( \mathfrak{P} \) denote its image in \( D(\Lambda) \). In [7] it is proved that there is a bijection, given by \( \mathfrak{P} \rightarrow \mathfrak{P} \cap A \), of the set of prime ideals of height one of \( \Lambda \) to the set of prime ideals of height one of \( A \). Let \( P(\Lambda) \) denote the set of prime ideals of \( \Lambda \).

Let \( M \in \mathfrak{P}(\Lambda) \). Then if \( p \) is a prime ideal of \( A \), the \( \Lambda_p \)-module \( M_p \) has finite length, denoted by \( l_p(M_p) \). Since \( M_p = 0 \) if \( M \in \mathfrak{P}(\Lambda) \), there is induced a map
\[ \chi: \mathfrak{P}/\mathfrak{P}(\Lambda) \rightarrow D(\Lambda) \]

defined by \( \chi(M) = \sum l_p(M_p) \text{ div } \mathfrak{P}, p = \mathfrak{P} \cap A, \mathfrak{P} \in P(\Lambda) \). The theorem will be proved if it can be shown that \( (D(\Lambda), \chi) \) satisfies the universal mapping property defining the Grothendieck group.
For a \( \Lambda \)-module \( M \), let \( \text{Ass} \ M \) denote the set of prime (two-sided) ideals \( \mathfrak{P} \) of \( \Lambda \) such that there is a nonzero submodule \( M' \) of \( M \) with \( \text{Ann}_\Lambda M'' = \mathfrak{P} \) for every nonzero submodule \( M'' \) of \( M' \) (see [7]).

**Proposition 3.** Let \( M \) be a finitely generated left \( \Lambda \)-module. Then there is a chain of submodules \( M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = 0, r \geq 0 \), such that \( M_i/M_{i+1} \) is isomorphic to a module \( \Lambda/\mathfrak{N}_i, \mathfrak{N}_i \) a left ideal of \( \Lambda \), where \( \text{Ass} \ \Lambda/\mathfrak{N}_i = \{ \mathfrak{P}_i \} \) and \( \text{Ann}_\Lambda (\Lambda/\mathfrak{N}_i) = \mathfrak{P}_i, \mathfrak{P}_i \) a prime ideal of \( \Lambda \).

The proof is the same as for Theorem 1 of [2, §1, n°4] and is omitted.

It is clear that \( \chi \) is additive on exact sequences, so Proposition 3 shows that

\[
\chi(M) = \sum_{i=0}^{r-1} \chi(\Lambda/\mathfrak{N}_i)
\]

where the \( \mathfrak{N}_i \) are left ideals satisfying the conclusion of Proposition 3.

The next proposition permits a study of these modules.

**Proposition 4.** Let \( \mathfrak{P} \subseteq \mathcal{P}(\Lambda), \mathfrak{p} = \mathfrak{P} \cap \Lambda \). Let \( m \) be a minimal left ideal in the simple \( \Lambda/\mathfrak{P}\mathfrak{p} \)-algebra \( \Lambda/\mathfrak{P}\mathfrak{p} \). Let \( n = m \cap (\Lambda/\mathfrak{P}) \). Then

(i) If \( \mathfrak{N} \) is a left ideal of \( \Lambda \) such that \( \text{Ass} \ \Lambda/\mathfrak{N} = \{ \mathfrak{P} \} \) and \( \mathfrak{N} \supseteq \mathfrak{P} \), then the class of \( \Lambda/\mathfrak{N} \) in \( G_t(\Lambda) \) is some integral multiple of the class of \( n \) in \( G_t(\Lambda) \).

(ii) \( \chi(n) = \text{div} \mathfrak{P} \).

**Proof.** Throughout this proof let \( S = \Lambda/\mathfrak{P} \). Let \([M]\) denote the class of \( M \) in \( G_t(\Lambda) \).

Let \( m_1 \) and \( m_2 \) be two minimal left ideals in \( S_\mathfrak{p} \). Then there is a \( t \) in \( S \), \( t \) a unit in \( S_\mathfrak{p} \), such that \( m_2 = m_1 t \). Let \( n_t = n_t \cap S \). Then \( n_t \subseteq n \), so consider the homomorphism \( n_1 \rightarrow n_2 \). When localized at \( \mathfrak{p} \) it is the isomorphism \( m_1 \rightarrow m_2 \). If \( q \) is a prime ideal of height one of \( \Lambda \) distinct from \( \mathfrak{p} \), then \( (n_1)_{\mathfrak{q}} = 0 = (n_2)_{\mathfrak{q}} \), so \( t \) localized at \( q \) is also an isomorphism. So in \( \mathfrak{P}(\Lambda) \) this map is an isomorphism, hence \([n_1] = [n_2] \).

Suppose that \( \mathfrak{N} \) is a left ideal satisfying the hypotheses of condition (i). Then \( (\mathfrak{N}/\mathfrak{P})_\mathfrak{p} \) is a left ideal in \( S_\mathfrak{p} \), so is the direct sum of minimal left ideals \( m_1, \cdots, m_t \) of \( S_\mathfrak{p} \). Let \( n_1 = n_1 \cap S \) and consider \( n_1 + \cdots + n_t \) in \( S \). This sum is direct. The homomorphisms \( n_1 + \cdots + n_t \rightarrow (\mathfrak{N}/\mathfrak{P})_\mathfrak{p} \cap S \) and \( \mathfrak{N}/\mathfrak{P} \rightarrow (\mathfrak{N}/\mathfrak{P})_\mathfrak{p} \cap S \) are isomorphisms at every localization. Hence \( t[n] = [n_1 + \cdots + n_t] = [\mathfrak{N}/\mathfrak{P}] \). This holds when \( \mathfrak{N} = \Lambda \), so let \([\Lambda/\mathfrak{P}] = [n_1 + \cdots + n_t] = s[n] \) where \( s = [\Lambda/\mathfrak{P}] : (\Lambda/\mathfrak{P})_\mathfrak{p} \). Then \( t \leq s \).

Now consider the exact sequence \( 0 \rightarrow \mathfrak{N}/\mathfrak{P} \rightarrow \Lambda/\mathfrak{P} \rightarrow \Lambda/\mathfrak{N} \rightarrow 0 \). Then
\[ \left[ \Lambda / \mathfrak{N} \right] = \left[ \Lambda / \mathfrak{P} \right] - \left[ \mathfrak{N} / \mathfrak{P} \right] \]
\[ = s[n] - t[n] \]
\[ = (s - t)[n]. \]

So (i) has been established. (ii) is clear from the definition of \( n \).

**Corollary.** For each \( \mathfrak{P} \in \mathcal{P}(\Lambda) \), let \( \pi(\mathfrak{P}) \) be a module constructed in Proposition 4. Then \( G_t(\Lambda) \) is free on the set \( \left[ \pi(\mathfrak{P}) \right] \).

This follows immediately from the two previous propositions.

**Proposition 5.** For each torsion left \( \Lambda \)-module \( M \), let \( g(M) \) be an element in an abelian group \( G \). Suppose \( g \) satisfies the two conditions

a. \( 0 \to M' \to M \to M'' \to 0 \) is an exact sequence in \( \mathfrak{S}(\Lambda) \), then \( g(M) = g(M') + g(M'') \).

b. \( M \in \mathfrak{P}(\Lambda) \), then \( g(M) = 0 \).

Then there is a unique homomorphism \( \theta : D(\Lambda) \to G \) such that \( g = \theta \chi \).

**Proof.** Let \( \pi(\mathfrak{P}) \) be an ideal of \( \Lambda / \mathfrak{P} \) defined in Proposition 4. Let \( \theta(\div \mathfrak{P}) = g(\pi(\mathfrak{P})) \). Then continue as in Proposition 11 of [3, §4, n°5]. Propositions 3 and 4 are designed to make that proof work.

Proposition 5 shows that \( D(\Lambda) \) satisfies the universal property which defines the Grothendieck group, so it must be isomorphic to it. This completes the proof of Theorem 2.

**Remark.** Since \( K^0(\Sigma) = K^0(D) = \mathbb{Z} \) in (HR) and \( \mathbb{Z} \) is \( \mathbb{Z} \) projective, \( G^0(\Gamma) = C(\Gamma) \oplus \mathbb{Z} \) where \( C(\Gamma) \) is the kernel of \( G^0(\Gamma) \to K^0(D) \), and hence is the image of \( G^0(\Gamma) \to G^0(\Gamma) \). A natural question is: What is an ideal (or module) theoretical description of the subgroup \( H \) of \( D(\Gamma) \) such that \( D(\Gamma)/H = C(\Gamma) ? \) \( C(\Gamma) \) is a generalization of the commutative class group (see [3]). A corollary to the corollary to Theorem 1 is that \( C(\Lambda) \) is isomorphic to \( C(\Gamma) \) and both do not depend on the maximal orders in question.

**Bibliography**