THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS

DONALD S. COHEN

We shall study the asymptotic behavior for $t \to \infty$ of solutions of the following nonlinear differential equation:

$$u'' + f(t, u) = 0.$$  

We suppose that $f(t, u)$ satisfies the following conditions:

H-1: $f(t, u)$ is continuous in $D: t \geq 0, -\infty < u < \infty$.

H-2: The derivative $f_u$ exists on $D$ and satisfies $f_u(t, u) > 0$ on $D$.

H-3: $|f(t, u(t))| \leq f_u(t, 0) |u(t)|$ on $D$.

An important class of functions $f(t, u)$ which satisfy conditions H-1, 2, 3 is the class of twice continuously differentiable functions $f(t, u)$ which are odd and strictly monotone in $u$ with $f_{uu} \geq 0$ for $u < 0$ and $f_{uu} \leq 0$ for $u > 0$. Nonlinear eigenvalue problems involving this class of functions have been studied extensively by G. H. Pimbley [1].

For the case $f(t, u) = \pm t^n u^n$, R. Bellman [2] has given an exhaustive treatment of the asymptotic behavior of proper solutions (i.e., solutions which exist and have continuous derivatives for $t \geq t_0$). For the case $f(t, u) = a(t)u^{2n+1}$ several results on asymptotic behavior exist depending on properties of $a(t)$. References can be found in the papers of P. Waltman [3] and R. A. Moore and Z. Nehari [4].

Our basic result is that there exist solutions of (1) which approach those of $u'' = 0$. More precisely, we prove the
THEOREM. Let \( f(t, u) \) satisfy H-1, 2, 3, and in addition, suppose that

\[
\int_1^\infty tf_u(t, 0)dt < \infty.
\]

Then, equation (1) has solutions which are asymptotic to \( a + bt \) as \( t \to \infty \), where \( b \neq 0 \).

PROOF. Our proof is essentially a modification of one given by Bellman [2, pp. 114–115] for the linear case \( f(t, u) = a(t)u \). Integrate (1) twice between 1 and \( t \) to obtain

\[
u(t) = c_1 + c_2t - \int_1^t (t - s)f(s, u(s))ds.
\]

From this we obtain, for \( t \geq 1 \),

\[
|u(t)| \leq (|c_1| + |c_2|)t + t\int_1^t |f(s, u(s))|ds.
\]

Using properties H-2, 3, we then obtain

\[
\frac{|u(t)|}{t} \leq (|c_1| + |c_2|) + \int_1^t sf_u(s, 0) \frac{|u(s)|}{s}ds.
\]

We now invoke the fundamental Gronwall inequality which states that if \( u, v \geq 0, c > 0 \) and if \( u(x) \leq c + \int_x^z u(\xi)v(\xi)d\xi \), then

\[
u(x) \leq c \exp\left(\int_0^x v(\xi)d\xi\right).
\]

Applying this to (4), we obtain

\[
\frac{|u(t)|}{t} \leq (|c_1| + |c_2|) \exp\left(\int_1^t sf_u(s, 0)ds\right).
\]

Finally, (2) and (5) imply

\[
|u(t)|/t \leq c_3.
\]

Now, by differentiating (3), we obtain

\[
u'(t) = c_2 - \int_1^t f(s, u(s))ds.
\]

Then, H-3 and (6) imply that

\[
\int_1^t |f(s, u(s))|ds \leq \int_1^t f_u(s, 0)u(s)ds \leq c_3 \int_1^t sf_u(s, 0)ds.
\]
Hence, as $t \to \infty$, the integral in (7) converges, and therefore $u'$ has a limit as $t \to \infty$. To ensure that this limit is not zero, we choose $c_2 = 1$ and use as a lower limit, instead of 1, a point $t_0$, where $t_0$ is chosen so that $c_3 \int_{t_0}^\infty s f_u(s, 0) \, ds < 1$.

Q.E.D.

REFERENCES


California Institute of Technology