I. Introduction. The theory of pseudoinverses, or generalized inverses, has been extensively developed over the last few years. A recent and comprehensive bibliography occurs in [2] which gives a short history also, and a brief survey of known results and computational methods.

We consider here the case of a bounded operator with closed range between Hilbert spaces. Most of the information for this case, which is used in this paper can be found in [3], which is pithy and short.

II. Notation and basic information. $A$ is a bounded linear transformation with closed range from $S_1$ to $S_2$, where $S_1$ and $S_2$ are Hilbert spaces with inner product norms $|\cdot|_1$ and $|\cdot|_2$, respectively. $A^*$ is the adjoint of $A$, $A^+$ is the pseudoinverse of $A$ (uniquely defined for this case [3]). $I_1$ and $I_2$ are the identity mappings on $S_1$ and $S_2$, respectively. $\overline{S}_1 = A^*S_2 = A^*S_1$ and $\overline{S}_2 = AS_1 = AA^*S_2$. We use a bar to denote the restriction of a mapping to $\overline{S}_1$ or $\overline{S}_2$, as appropriate, e.g., $\overline{A} = A|_{\overline{S}_1}$ the restriction of $A$ to $\overline{S}_1$. We know that $(A)^{-1}$ exists and is bounded [3]; and it is readily deduced that $|A^+| = |(A)^{-1}|$ (from [3, Theorem 3], noting that $\overline{S}_1 = \mathfrak{R}(A)^\perp$, and $\overline{A}^+ - (A)^{-1} = 0$). Thus, we know a priori that $A^+$ is a uniquely defined, bounded linear transformation from $S_2$ to $S_1$. We use the conventional norms on linear transformations. We use the basic definition of the pseudoinverse given by Desoer and Whalen [3], for the case studied $B = A^+$ if and only if $B$ is a linear operator such that

(i) $BA = I_1$,

(ii) $BS_2^\perp = 0$ (i.e., $By = 0$ if $y$ is in the orthogonal complement of $S_2$).

From this definition it follows that $AA^+A = A$ and $A^*AA^+ = A^*$, since $S_1 = \overline{S}_1 \oplus \overline{S}_1^\perp$ and $S_2 = \overline{S}_2 \oplus \overline{S}_2^\perp$.

III. Theorem 1. Define, for $t \geq 0$,

(3.1) \[ A^+(t) = \int_0^t \exp[-A^*A(t-s)]A^*ds. \]

Received by the editors July 7, 1966.

1 This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, with the National Science Foundation, Grant NSF-GP-3465.
Then
\[
|A^+ - A^+(t)| \leq |A^+| \exp[-t|A^+|^2], \quad \forall t.
\]

**Proof.** Since \(A^+ = A^*A A^+, (3.1)\) may be evaluated as: \(A^+(t) = [I_1 - \exp(-tA^*A)]A^+.\) Pick \(y \in S_2,\) and let \(\alpha(t) = [A^+ - A^+(t)]y = \exp(-tA^*A)A^+y.\) Clearly, \(\alpha(t) \in \mathbb{S}_1, \forall t.\) Thus \(d|\alpha(t)|^2/dt = -2|A\alpha(t)|^2 (\text{see} [5]) \leq -2|A^+|^2|\alpha(t)|^2, \) whence \(|A^+y - A^+(t)y| \leq |A^+y|\exp[-t|A^+|^2],\) and (3.2) follows since \(y\) is arbitrary.

The next theorem has been proved for the matrix case by den Broeder and Charnes (see [2]), and for the nonsingular case by Altman [1].

**Theorem 2.** Pick \(c\) so that \(0 < c < 2, \forall n \geq 0,\) define

\[
B_{n,c} = \sum_{p=0}^{n} \left[ I_1 - \frac{cA^*A}{|A|^2} \right] p \frac{cA^*}{|A|^2}.
\]

Define:
\[
\beta_c = \max \{ |1 - c|, |1 - c/|A|^2|A^+|^2| \} < 1.
\]

Then
\[
|A^+ - B_{n,c}| \leq (c\beta_c^{n+1}/(1 - \beta_c)) |A|, \quad \forall n \geq 0.
\]

**Proof of Theorem 2.** Let \(B_{n,c} = \lim_{n \to \infty} B_{n,c},\) and define \(E = cA^*A/|A|^2.\) Then \(|I_1 - E| \leq \beta_c (\text{see} [1, pp. 52-55])\) so that \((E)^{-1} = \sum_{p=0}^{\infty} (I_1 - E)^p (\text{where} (E)^{-1} \text{is defined on} \mathbb{S}_1, \text{the range of} E).\) Now \(E: \mathbb{S}_1 \to \mathbb{S}_1 \) and \((I_1 - E): \mathbb{S}_1 \to \mathbb{S}_1,\) so
\[
B_{n,c}A = \sum_{p=0}^{\infty} (I_1 - E)^p E
\]= \sum_{p=0}^{\infty} (I_1 - E)^p \frac{E}{p} = T_1.
\]

\(B_{n,c}S_2 = 0, \forall n,\) since \(A^*S_2 = 0,\) in view of (3.3). Linearity is obvious. Thus \(A^+ = B_{n,c};\) and (3.4) follows immediately, since it is the rate of convergence on \(S_2\) (see [1, p. 52-55]). Optimal convergence occurs when
\[
c = 2 |A|^2 |A^+|^2/(|A|^2 |A^+|^2 + 1),
\]
in which case
\[
\beta_c = (|A|^2 |A^+|^2 - 1)/(|A|^2 |A^+|^2 + 1) \quad [1],
\]
which yields the following
Corollary to Theorem 2. If $|A| = |A^\dagger| = 1$, then $A^\dagger = A^\ast$.

The rates of convergence in the previous theorems are often impractically slow. The following theorem gives a rapidly convergent recursion sequence, analogous to a well-known technique for improving estimates of the inverse for a nonsingular matrix [4, p. 120].

Theorem 3. Pick $\epsilon$ and define $\beta$ as in Theorem 2. Define the sequence:

$$D_{0,c} = \epsilon A^\ast / \|A\|^2; \quad D_{n+1,c} = 2D_{n,c} - D_{n,c}AD_{n,c}.$$

Then

$$\|A^\dagger - D_{n,c}\| \leq \|A\| \cdot \beta_e^{2^n}.$$ (3.6)

Proof of Theorem 3. Define $\Gamma_c = [A^\dagger A - D_{0,c}A]$. $\Gamma_c \bar{S}_1^c = 0$ and $\bar{I}_1 - \bar{E}$ (see proof of Theorem 2) so $|\Gamma_c| \leq \beta_c$. By definition $\Gamma_c^{2^n} = [A^\dagger A - D_{0,c}A]$. Suppose that $\Gamma_c^{2^n} = [A^\dagger A - D_{n,c}A]$. Then

$$\Gamma_c^{2^n+1} = (\Gamma_c^{2^n})^2 = A^\dagger AA^\dagger A - A^\dagger AD_{n,c}A$$

$$- D_{n,c}A A^\dagger A + D_{n,c}AD_{n,c}A$$

$$= A^\dagger A - (2D_{n,c} - D_{n,c}AD_{n,c})A$$

$$= [A^\dagger A - D_{n+1,c}A],$$

since $D_{n,c}A : \mathbb{S}_1 \rightarrow \mathbb{S}_1, \forall n$, which is clear from (3.5), and since $AA^\dagger A = A$. Also from (3.5), $D_{n,c}S_2^c = 0 = A^\dagger S_2^c, \forall n$. Since $|\Gamma_c| \leq \beta_c$, and $A^\dagger - D_{n,c} = (A^\dagger - D_{n,c}) AA^\dagger, \forall n$, (3.6) follows.

IV. So long as $|A^\dagger| < \infty$, the hypothesis that $|A| < \infty$ may be relaxed in Theorem 1. We need only require that $A^\ast A$ is a closed mapping with dense domain. The proof as given remains valid (see [5]).

If $A$ fails to have closed range, but is bounded, then $A^\dagger(t), B_{n,c}$, and $D_{n,c}$ converge monotonically to $A^\dagger$, but (3.2), (3.4), (3.6) are useless since $|A^\dagger| = \infty$ and $\beta_c = 1$. The associated proofs are more lengthy and will be published elsewhere.

References


Courant Institute of Mathematical Sciences, New York University