SHORTER NOTES

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THE MORSE INDEX THEOREM

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The use of a matrix Riccati equation to establish sufficiency theorems in the calculus of variations is well known (see [3], e.g.). In this note we extend the method to give an elementary proof of the Morse index theorem, thereby eliminating the ad hoc subdivisions of the classical proof (see [2], [4], or [5], e.g.). Although we consider only the simplest form of the index theorem, the technique can easily be adapted to generalizations such as that of [1].

Let E be a euclidean space, G the linear space of broken C^∞ maps u: $[0, T]\to E$ for some closed interval $[0, T]$, and H the subspace of those $u\in G$ such that $u(0)=u(T)=0$. Let P be a given C^∞ map of $[0, T]$ into the selfadjoint linear transformations of E, and let U: $[0, T]\to \mathrm{Hom}_R(E, E)$ be the unique C^∞ map satisfying $U(0)=0$, $U'(0)=I$, and $U''+PU=0$; U can also be regarded as a linear transformation of G into itself which is stable on the subspace H. The *multiplicity* of any $t\in(0, T]$ is the nullity of $U(t)$, and t is a *focal point* whenever it has positive multiplicity. The *index form* I on $H\times H$ is defined by $I(u, v)=\int_{t=0}^{T}[(u', v')-(Pu, v)]dt$, and an inner product J on $H\times H$ is defined by $J(u, v)=\int_{t=0}^{T}(u, v)dt$, where $(\ ,\ )$ is in both cases the inner product on $E\times E$; for convenience we set $I(u, u)=I(u)$ and $J(u, u)=J(u)$. The *index* $i(I)$ is the dimension of any maximal subspace of H on which I is negative definite, and the *nullity* $n(I)$ is the dimension of the subspace of those $u\in H$ such that $I(u, v)=0$ for all $v\in H$.

The first portion of our proof of the index theorem is standard, depending on the following classical lemma:

LEMMA. *If I/J is minimized by some nonzero $u\in H$ then u is everywhere C^∞.*

PROOF. For any $v\in H$ the quotient $I(u+\epsilon v)/J(u+\epsilon v)$ assumes a minimum value at $\epsilon=0$, so that for $\lambda=I(u)/J(u)$ an integration by parts gives

Received by the editors March 17, 1967.

[1] Research supported by National Science Foundation Grant NSF-GP-5477.

$$\int_{t=0}^{T}\left(u'(t)+\int_{s=0}^{t}(\lambda u+Pu)ds,\ v'(t)\right)dt = I(u, v) - \lambda J(u, v) = 0.$$

Hence $u'(t)+\int_{s=0}^{t}\lambda u+Pu\,ds$ is independent of t by the fundamental lemma of the calculus of variations, which implies the desired result since P is C^∞.

MORSE INDEX THEOREM. *The index $i(I)$ is the sum of the multiplicities at all focal points in the open interval* $(0, T)$, *and the nullity $n(I)$ is the multiplicity of T.*

PROOF. Let r be the sum of the multiplicities at focal points $t_1, \cdots, t_r \in (0, T)$, where $0<t_1 \leqq \cdots \leqq t_r < T$ with equalities permitted. Since the nullity of $U(t_1)$ is positive there exists a nonzero C^∞ map u: $[0, t_1]\to E$ satisfying $u(0)=u(t_1)=0$ and

$$(*) \qquad u''+Pu=0,$$

and one extends u to $u_1 \in H$ by setting $u_1(t)=0$ for $t\in(t_1, T]$. An integration by parts establishes $I(u_1)=0$, and since u_1 has a discontinuous derivative at t_1 the Lemma assures the existence of a $v_1\in H$ satisfying $I(v_1)<0$; replacing v_1 by $-v_1$ if necessary we may assume $I(u_1, v_1)\leqq 0$, hence $I(v_1+\epsilon u_1)<0$ for all $\epsilon\geqq 0$. Since the nullity of $U(t_2)$ is positive one similarly finds a nontrivial $u_2\in H$ with discontinuous derivative at t_2 such that $I(u_2)=0$, hence a $v_2\in H$ such that $I(v_2+\epsilon u_2)<0$ for all $\epsilon\geqq 0$; if $t_2=t_1$ then the nullity of $U(t_1)$ is at least 2 so that u_2 can be chosen to be linearly independent of u_1, and if $t_2>t_1$ then u_2 is automatically linearly independent of u_1 since u_2 cannot vanish identically in (t_1, t_2). Continuing in this fashion one finds $u_1, \cdots, u_r, v_1, \cdots, v_r\in H$ such that $u_1, \cdots, u_r$ are linearly independent and $I(v_i+\epsilon u_i)<0$ for all $\epsilon\geqq 0$. For sufficiently large ϵ the functions $v_1+\epsilon u_1, \cdots, v_r+\epsilon u_r$ are linearly independent, and one may therefore suppose $v_1, \cdots, v_r$ to be themselves linearly independent. Let H^- denote the r-dimensional subspace of H spanned by $v_1, \cdots, v_r$, observing that I is negative definite on H^-.

Let ${}^tU$ represent the transpose of U with respect to (,) at all points of $[0, T]$, and note that $({}^tUU')' = {}^tU'U' + {}^tUU'' = {}^tU'U' - {}^tUPU$, which is selfadjoint at all points of $[0, T]$. Since $({}^tUU')(0) = 0$ it follows that ${}^tUU'$ is everywhere selfadjoint, hence one obtains a selfadjoint solution V of the Riccati equation $V'+V^2+P=0$ on $(0, T]\sim\{\text{focal points}\}$ by setting $V=U'U^{-1}={}^tU^{-1}({}^tUU')U^{-1}$. Let H^+ denote the intersection of H with the image of G under U, and observe that since U is nonsingular on a dense subset of $[0, T]$ and the elements of G are continuous V may be regarded as a linear map

from H^+ to G. Then $(Vu, u)' = 2(Vu, u') - (Vu, Vu) - (Pu, u)$ for any $u \in H^+$, hence $(u', u') - (Pu, u) = (u' - Vu, u' - Vu) + (Vu, u)'$; since (Vu, u) vanishes at 0 and T it follows that

$$I(u) = \int_{t=0}^{T} (u' - Vu, u' - Vu)dt,$$

consequently that I is positive semidefinite on H^+.

To prove the first assertion of the theorem it remains only to establish that H is the internal direct sum of H^- and H^+. The requirement $H^- \cap H^+ = \{0\}$ is automatically satisfied since I is negative definite on H^- and positive semidefinite on H^+, but to prove $H^- + H^+ = H$ we need an alternate characterization of H^+. Since U is nonsingular at all points of $(0, T]$ except the focal points, H^+ consists of those $u \in H$ such that $u(t_i)$ lies in the image of $U(t_i)$ for each focal point t_i; hence if $\theta_1, \cdots, \theta_r$ are elements of $\mathrm{Hom}_R(E, R)$ such that $\theta_1 U(t_1) = \cdots = \theta_r U(t_r) = 0 \in \mathrm{Hom}_R(E, R)$, where $\theta_{i+1}, \cdots, \theta_{i+j}$ are linearly independent whenever $t_{i+1} = \cdots = t_{i+j}$, H^+ is the subspace of those $u \in H$ such that $\langle \theta_1, u(t_1) \rangle = \cdots \langle \theta_r, u(t_r) \rangle = 0$.

If $\sum_i \langle \theta_j, v_i(t_j) \rangle \alpha_i = 0$ for real numbers $\alpha_1, \cdots, \alpha_r$, where $j = 1, \cdots, r$, then $\sum_i \alpha_i v_i$ is simultaneously a member of H^- and H^+ so that $\alpha_1 = \cdots = \alpha_r = 0$; consequently $\det(\langle \theta_j, v_i(t_j) \rangle) \neq 0$ and hence for any $u \in H$ there are unique $\alpha_1, \cdots, \alpha_r$ such that $\sum_i \langle \theta_j, v_i(t_j) \rangle \alpha_i = \langle \theta_j, u(t_j) \rangle$ for all j, for which $u = \sum_i \alpha_i v_i + (u - \sum_i \alpha_i v_i) \in H^- + H^+$ as desired.

Finally let H^0 denote the subspace of those $u \in H$ such that $I(u, v) = 0$ for all $v \in H$, and for any $u \in H^0$ let $u^- + u^+$ be the preceding direct sum decomposition. Then $I(u, u^- - u^+) = I(u^-) - I(u^+) < 0$ except when $u^- = 0$ and $I(u^+) = 0$, so that H^0 is a subspace of elements $u \in H^+$ satisfying $I(u) = 0$. Conversely, if $I(u) = 0$ for some $u \in H^+$ then $\int_{t=0}^{T} (u' - Vu, u' - Vu)dt = 0$, hence $u' = Vu = U'U^{-1}u$, which is equivalent to $(U^{-1}u)' = 0$, that is, $u = Uc$ for some constant map c: $[0, T] \to E$. This happens if and only if u is a solution of (*) such that both $u(0) = 0$ and $u(T) = 0$, and for any $v \in H$ an integration by parts therefore gives

$$I(u, v) = \int_{t=0}^{T} (u', v') - (Pu, v)dt = \int_{t=0}^{T} (-u'' - Pu, v)dt = 0,$$

hence $u \in H^0$. Consequently the final assertion of the index theorem is established by the remark that the multiplicity of T is exactly the dimension of the subspace of those solutions of (*) such that both $u(0) = 0$ and $u(T) = 0$.

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CONVOLUTION OF L^p FUNCTIONS

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Rajagopalan and Zelazko [1] and [2] have proved that if G is a noncompact, locally compact group, $L^p(G)$ is not closed under convolution for $p>2$. In fact, although they did not prove it, the convolution of two L^p functions need not exist. This is a special case of the following:

THEOREM. *If $1<p<\infty$, $1<r<\infty$, and $1/p+1/r<1$, and if G is a noncompact locally compact group, then there is an open set U in G, aud there are functions $f\in L^p(G)$, $g\in L^r(G)$, such that $f*g(y)$ is not defined for y in U.*

PROOF. Let H be the subgroup of G consisting of those elements on which the modular function is 1. It is easily proved that H is closed and noncompact. Let V be a compact symmetric neighbourhood of the identity in G, and set $W=V\cdot V$. We can inductively choose a sequence x_n of elements of H such that $Wx_i\cap Wx_j=\emptyset$ for $i\neq j$ (see [2]). There is an open subset U of V such that for y in U, $y^{-1}V\cap V$ is a set of positive measure. Define $\rho>1$ so that $\rho(1/p+1/r)=1$. Choose $\epsilon_n=\pm 1$, so that the partial sums of the series $\sum\epsilon_n/n$ are everywhere dense in the real line. Define the function f on G by $f(xx_n)=1/n^{\rho/p}$ for x in V, and extend f to G by defining it to be zero outside $\bigcup Vx_n$. Likewise define h so that $h(xx_n)=\epsilon_n/n^{\rho/r}$ for x in V. Define $g(y)=h(y^{-1})$. Evidently f is in L^p and h in L^r. Since the mod-

Received by the editors March 5, 1967.

[1] Supported by National Science Foundation grant GP-5803.