SHORTER NOTES

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THE MORSE INDEX THEOREM

HOWARD OSBORN

The use of a matrix Riccati equation to establish sufficiency theorems in the calculus of variations is well known (see [3], e.g.). In this note we extend the method to give an elementary proof of the Morse index theorem, thereby eliminating the ad hoc subdivisions of the classical proof (see [2], [4], or [5], e.g.). Although we consider only the simplest form of the index theorem, the technique can easily be adapted to generalizations such as that of [1].

Let $E$ be a euclidean space, $G$ the linear space of broken $C^\infty$ maps $u: [0, T] \to E$ for some closed interval $[0, T]$, and $H$ the subspace of those $u \in G$ such that $u(0) = u(T) = 0$. Let $P$ be a given $C^\infty$ map of $[0, T]$ into the selfadjoint linear transformations of $E$, and let $U: [0, T] \to \text{Hom}_R(E, E)$ be the unique $C^\infty$ map satisfying $U(0) = 0$, $U''(0) = I$, and $U''+PU = 0$; $U$ can also be regarded as a linear transformation of $G$ into itself which is stable on the subspace $H$. The multiplicity of any $t \in (0, T]$ is the nullity of $U(t)$, and $t$ is a focal point whenever it has positive multiplicity. The index form $I$ on $H \times H$ is defined by $I(u, v) = \int_{t=0}^{T} [(u', v') - (Pu, v)]dt$, and an inner product $J$ on $H \times H$ is defined by $J(u, v) = \int_{t=0}^{T} (u, v)dt$, where $(\ , \ )$ is in both cases the inner product on $E \times E$; for convenience we set $I(u, u) = I(u)$ and $J(u, u) = J(u)$. The index $i(I)$ is the dimension of any maximal subspace of $H$ on which $I$ is negative definite, and the nullity $n(I)$ is the dimension of the subspace of those $u \in H$ such that $I(u, v) = 0$ for all $v \in H$.

The first portion of our proof of the index theorem is standard, depending on the following classical lemma:

**Lemma.** If $I/J$ is minimized by some nonzero $u \in H$ then $u$ is everywhere $C^\infty$.

**Proof.** For any $v \in H$ the quotient $I(u+\epsilon v)/J(u+\epsilon v)$ assumes a minimum value at $\epsilon = 0$, so that for $\lambda = I(u)/J(u)$ an integration by parts gives

Received by the editors March 17, 1967.

1 Research supported by National Science Foundation Grant NSF-GP-5477.
\[
\int_{t=0}^{T} \left( u'(t) + \int_{s=0}^{t} (\lambda u + Pu) ds, v'(t) \right) dt = I(u, v) - \lambda J(u, v) = 0.
\]

Hence \( u'(t) + \int_{s=0}^{t} \lambda u + Pu \, ds \) is independent of \( t \) by the fundamental lemma of the calculus of variations, which implies the desired result since \( P \) is \( C^\infty \).

**Morse Index Theorem.** The index \( i(I) \) is the sum of the multiplicities at all focal points in the open interval \((0, T)\), and the nullity \( n(I) \) is the multiplicity of \( T \).

**Proof.** Let \( r \) be the sum of the multiplicities at focal points \( t_1, \ldots, t_r \in (0, T) \), where \( 0 < t_1 \leq \cdots \leq t_r < T \) with equalities permitted. Since the nullity of \( U(t_i) \) is positive there exists a nonzero \( C^\infty \) map \( u : [0, t_1] \to E \) satisfying \( u(0) = u(t_1) = 0 \) and

\[
(*) \quad u'' + Pu = 0,
\]

and one extends \( u \) to \( u_1 \in H \) by setting \( u_1(t) = 0 \) for \( t \in (t_1, T] \). An integration by parts establishes \( I(u_1) = 0 \), and since \( u_1 \) has a discontinuous derivative at \( t_1 \) the Lemma assures the existence of a \( v_1 \in H \) satisfying \( I(v_1) < 0 \); replacing \( v_1 \) by \(-v_1\) if necessary we may assume \( I(u_1 + \epsilon v_1) \leq 0 \) for all \( \epsilon \geq 0 \). Since \( I(v_1) < 0 \) for all \( \epsilon \geq 0 \). Since the nullity of \( U(t_2) \) is positive one similarly finds a nontrivial \( u_2 \in H \) with discontinuous derivative at \( t_2 \) such that \( I(u_2) = 0 \), hence a \( v_2 \in H \) such that \( I(v_2 + \epsilon u_2) < 0 \) for all \( \epsilon \geq 0 \); if \( t_2 = t_1 \) then the nullity of \( U(t_1) \) is at least 2 so that \( u_2 \) can be chosen to be linearly independent of \( u_1 \), and if \( t_2 > t_1 \) then \( u_2 \) is automatically linearly independent of \( u_1 \) since \( u_2 \) cannot vanish identically in \( (t_1, t_2) \). Continuing in this fashion one finds \( u_1, \ldots, u_r, v_1, \ldots, v_r \in H \) such that \( u_1, \ldots, u_r \) are linearly independent and \( I(v_1 + \epsilon u_1, \ldots, v_r + \epsilon u_r) < 0 \) for all \( \epsilon \geq 0 \). For sufficiently large \( \epsilon \) the functions \( v_1 + \epsilon u_1, \ldots, v_r + \epsilon u_r \) are linearly independent, and one may therefore suppose \( v_1, \ldots, v_r \) to be themselves linearly independent. Let \( H^- \) denote the \( r \)-dimensional subspace of \( H \) spanned by \( v_1, \ldots, v_r \), observing that \( I \) is negative definite on \( H^- \).

Let \( ^tU \) represent the transpose of \( U \) with respect to \( (, , ) \) at all points of \([0, T]\), and note that \( (\ ^tUU') = ^tU'U' + (\ ^tUU'', \ ^tUU') = ^tU'U' - ^tUPUU^t \), which is selfadjoint at all points of \([0, T]\). Since \( (\ ^tUU')(0) = 0 \) it follows that \( ^tUU' \) is everywhere selfadjoint, hence one obtains a selfadjoint solution \( V \) of the Riccati equation \( V' + V^2 + P = 0 \) on \((0, T) - \{\text{focal points}\} \) by setting \( V = U'U^{-1} = UU^{-1}(\ ^tUU')U^{-1} \). Let \( H^+ \) denote the intersection of \( H \) with the image of \( G \) under \( U \), and observe that since \( U \) is nonsingular on a dense subset of \([0, T]\) and the elements of \( G \) are continuous \( V \) may be regarded as a linear map.
from $H^+ \to G$. Then $(Vu, u') = 2(Vu, u') - (Pu, u) - (Pu, u')$ for any $u \in H^+$, hence $(u', u') - (Pu, u) = (u' - Vu, u' - Vu) + (Vu, u')$; since $(Vu, u)$ vanishes at 0 and $T$ it follows that

$$I(u) = \int_{t=0}^{T} (u' - Vu, u' - Vu)dt,$$

consequently that $I$ is positive semidefinite on $H^+$.

To prove the first assertion of the theorem it remains only to establish that $H$ is the internal direct sum of $H^-$ and $H^+$. The requirement $H^- \cap H^+ = \{0\}$ is automatically satisfied since $I$ is negative definite on $H^-$ and positive semidefinite on $H^+$, but to prove $H^- + H^+ = H$ we need an alternate characterization of $H^+$. Since $U$ is nonsingular at all points of $(0, T]$ except the focal points, $H^+$ consists of those $u \in H$ such that $u(t_i)$ lies in the image of $U(t_i)$ for each focal point $t_i$; hence if $\theta_1, \ldots, \theta_r$ are elements of $\text{Hom}_R(E, R)$ such that $\theta_1 U(t_1) = \cdots = \theta_r U(t_r) = 0 \in \text{Hom}_R(E, R)$, where $\theta_{i+1}, \ldots, \theta_{i+j}$ are linearly independent whenever $t_{i+1} = \cdots = t_{i+j}$, $H^+$ is the subspace of those $u \in H$ such that $\langle \theta_j, u(t_j) \rangle = \cdots = \langle \theta_r, u(t_r) \rangle = 0$.

If $\sum_i \langle \theta_j, v_i(t_j) \rangle \alpha_i = 0$ for real numbers $\alpha_1, \ldots, \alpha_r$, where $j = 1, \ldots, r$, then $\sum_i \alpha_i v_i$ is simultaneously a member of $H^-$ and $H^+$ so that $\alpha_1 = \cdots = \alpha_r = 0$; consequently $\det \langle \langle \theta_j, v_i(t_j) \rangle \rangle \neq 0$ and hence for any $u \in H$ there are unique $\alpha_1, \ldots, \alpha_r$ such that $\sum_i \langle \theta_j, v_i(t_j) \rangle \alpha_i = \langle \theta_j, u(t_j) \rangle$ for all $j$, for which $u = \sum_i \alpha_i v_i + (u - \sum_i \alpha_i v_i) \in H^- + H^+$ as desired.

Finally let $H^0$ denote the subspace of those $u \in H$ such that $I(u, v) = 0$ for all $v \in H$, and for any $u \in H^0$ let $u^- + u^+$ be the preceding direct sum decomposition. Then $I(u, u^- - u^+) = I(u^-) - I(u^+) < 0$ except when $u^- = 0$ and $I(u^+) = 0$, so that $H^0$ is a subspace of elements $u \in H^+$ satisfying $I(u) = 0$. Conversely, if $I(u) = 0$ for some $u \in H^+$ then $\int_{t=0}^{T} (u' - Vu, u' - Vu)dt = 0$, hence $u' = Vu = U'U^{-1}u$, which is equivalent to $(U^{-1}u)' = 0$, that is, $u = Uc$ for some constant map $c : [0, T] \to E$. This happens if and only if $u$ is a solution of (§) such that both $u(0) = 0$ and $u(T) = 0$, and for any $v \in H$ an integration by parts therefore gives

$$I(u, v) = \int_{t=0}^{T} (u', v') - (Pu, v)dt = \int_{t=0}^{T} (-u'' - Pu, v)dt = 0,$$

hence $u \in H^0$. Consequently the final assertion of the index theorem is established by the remark that the multiplicity of $T$ is exactly the dimension of the subspace of those solutions of (§) such that both $u(0) = 0$ and $u(T) = 0$. 

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CONVOLUTION OF $L^p$ FUNCTIONS

NEIL W. RICKERT

Rajagopalan and Zelazko [1] and [2] have proved that if $G$ is a noncompact, locally compact group, $L^p(G)$ is not closed under convolution for $p > 2$. In fact, although they did not prove it, the convolution of two $L^p$ functions need not exist. This is a special case of the following:

**Theorem.** If $1 < p < \infty$, $1 < r < \infty$, and $1/p + 1/r < 1$, and if $G$ is a noncompact locally compact group, then there is an open set $U$ in $G$, and there are functions $f \in L^p(G)$, $g \in L^r(G)$, such that $f \ast g(y)$ is not defined for $y$ in $U$.

**Proof.** Let $H$ be the subgroup of $G$ consisting of those elements on which the modular function is 1. It is easily proved that $H$ is closed and noncompact. Let $V$ be a compact symmetric neighbourhood of the identity in $G$, and set $W = V \cdot V$. We can inductively choose a sequence $x_n$ of elements of $H$ such that $Wx_i \cap Wx_j = \emptyset$ for $i \neq j$ (see [2]). There is an open subset $U$ of $V$ such that for $y$ in $U$, $y^{-1}V \cap V$ is a set of positive measure. Define $\rho > 1$ so that $\rho(1/p + 1/r) = 1$. Choose $\epsilon_n = \pm 1$, so that the partial sums of the series $\sum \epsilon_n/n$ are everywhere dense in the real line. Define the function $f$ on $G$ by $f(xx_n) = 1/n^{p/r}$ for $x$ in $V$, and extend $f$ to $G$ by defining it to be zero outside $Vx_n$. Likewise define $h$ so that $h(xx_n) = \epsilon_n/n^{p/r}$ for $x$ in $V$. Define $g(y) = h(y^{-1})$. Evidently $f$ is in $L^p$ and $h$ in $L^r$. Since the mod-

Received by the editors March 5, 1967.

1 Supported by National Science Foundation grant GP-5803.