ON p-ABELIAN GROUPS

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Finite p-groups in which the relation $(xy)^p = x^p y^p$ is satisfied by all elements x, y have been called *p-abelian* by C. Hobby [1]. Two examples of classes of such groups are the groups of exponent p and the abelian p-groups.

The purpose of this note is to show that these two classes of groups "span" the class of p-abelian groups in the following sense. If G is a p-abelian group, then there is a finite group P of exponent p, and a finite abelian p-group, A such that G is a factor group of a subgroup of $P \times A$.

Since the property of being p-abelian is defined by an identical relation, the class of those groups which satisfy this relation forms a variety. (The p-abelian groups form a proper subset of this variety.) This result can then be restated in varietal terms. We use the notation var G to denote the smallest variety containing the group G. By $\mathfrak{A} \cup \mathfrak{B}$ we denote the smallest variety containing the varieties $\mathfrak{A}$ and $\mathfrak{B}$ as subclasses.

THEOREM. *Let G be a p-abelian group of exponent p^α, $\alpha \geqq 1$. Then* var $G =$ var $P \cup$ var C_{p^α} *with P a finite group of exponent p and C_{p^α} the cyclic group of order p^α.*

Conversely, if P is a finite group of exponent p and A is a finite abelian p-group, then the finite groups in var $P \cup$ var A *are p-abelian.*

The proof of this theorem depends on Theorem 1.3 of [2] which we state below without proof as "Proposition." First we need to recall the definition of regularity and prove a few elementary lemmas.

DEFINITION 1. A finite p-group is called *regular* if for all $a, b \in G$, $(ab)^p = a^p b^p c^p$ with c an element in the derived group of the group generated by a and b.

We have as an immediate consequence

LEMMA 1. *If G is p-abelian, then G is regular.*

LEMMA 2. *If G is a nonabelian p-abelian group, then G' (the derived group of G) has exponent p.*

PROOF. Since G is p-abelian it follows that a^p is central in G for all

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$a \in G$ (Theorem 1 (1.2) of [1]). Thus $(a^p, b)=1$ for all $a, b \in G$. But since G is regular it follows that $(a, b)^p=1$ for all $a, b \in G$. Thus if $G' \neq 1$, G' has exponent p.

REMARK. Clearly, if G is a regular p-group for which $(G')^p=1$, then G is p-abelian. Thus p-abelian groups may be characterized by the conditions: (1) G is regular, (2) $(G')^p=1$.

Before we state the Proposition we need to recall the definition of a basic group.

DEFINITION 2. A group G is called *basic* if G is critical (G is not contained in var $\{A_\alpha\}$ where $\{A_\alpha\}$ is the set of proper subgroups and factor groups of G) and var G is not the join of proper subvarieties.

Basic groups derive their importance from the fact that if G is a finite group, then var G is the join of $\{\text{var } H_\alpha\}$ where H_α are the finite basic groups in var G. Furthermore the number of basic groups in var G is finite and if the basic groups in var G all satisfy some identical relation, then G does as well.

PROPOSITION. *Let G be a finite regular basic p-group. Then the exponent of G equals the exponent of G'.*

PROOF OF THEOREM. Since G is a finite p-group, it has some exponent, say p^α. Thus $C_{p^\alpha} \in \text{var } G$ and all abelian basic groups in var G will have exponent dividing p^α. Hence C_{p^α} will generate all abelian basic groups in var G.

If H is a nonabelian basic group in var G, then since H is p-abelian it is regular with H' of exponent p. Then it follows from the proposition that H has exponent p. Thus the finitely many basic nonabelian finite groups in var G are of exponent p and their direct product P, is a finite group of exponent p. Hence var $G=\text{var } P \cup \text{var } C_{p^\alpha}$.

Conversely, since P and A both satisfy the identical relation $(xy)^p = x^p y^p$, then every group G in var $P \cup$ var A satisfies it as well. Now since P and A are finite p-groups, it follows that if G is finite, then G is a p-group and thus G is p-abelian.

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