A TRANSFORMATION THEOREM ON
SPECTRAL MEASURES

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1. Introduction. Let $\mathcal{B}$ be a $\sigma$-algebra of subsets of a given space $\Omega$, and let $E$ be a countably additive projection-valued measure for a Hilbert space $\mathcal{H}$, i.e., for each $B \in \mathcal{B}$, $E(B)$ is a projection operator on $\mathcal{H}$ such that $E(\bigcup_k B_k) = \sum_k E(B_k)$ whenever $\{B_k\}$ is a sequence of disjoint sets in $\mathcal{B}$ and $E(\Omega) = I$, where $I$ denotes the identity operator on $\mathcal{H}$. We will refer to $(\Omega, \mathcal{B}, \mathcal{H}, E)$ as a spectral measure for the Hilbert space $\mathcal{H}$.

For each element $x$ in $\mathcal{H}$ the function $|E(\cdot)x|^2$ is a countably additive nonnegative measure on $\mathcal{B}$ such that $|E(\Omega)x|^2 = |x|^2 < \infty$. If $\phi$ is a $\mathcal{B}$-measurable function defined on $\Omega$, then the integral $\int_{\Omega} \phi(\omega) E(d\omega)$ is a well-defined linear operator $T$ (not necessarily bounded) in $\mathcal{H}$ with domain

$$\mathcal{D}_T = \left\{ x: x \in \mathcal{H} \& \int_{\Omega} |\phi(\omega)|^2 |E(d\omega)x|^2 < \infty \right\}.$$ 

It is a simple result that if $x \in \mathcal{D}_T$, then

$$|Tx|^2 = \int_{\Omega} |\phi(\omega)|^2 |E(d\omega)x|^2.$$ 

In this paper we propose to establish the following theorem concerning transformation of spectral measures for a Hilbert space $\mathcal{H}$.

2. Theorem. Let

(i) $(\Omega_1, \mathcal{B}_1, \mathcal{H}, E_1)$ be a spectral measure for a Hilbert space $\mathcal{H}$,
(ii) $T$ be a single-valued transformation on $\Omega_1$ into a space $\Omega_2$,
(iii) $\mathcal{B}_2 = \{ B: B \subseteq \Omega_2 \text{ and } T^{-1}(B) \in \mathcal{B}_1 \}$,
(iv) for each $B \in \mathcal{B}_2$,

$$E_2(B) = E_1(T^{-1}(B)).$$

Then (a) $(\Omega_2, \mathcal{B}_2, \mathcal{H}, E_2)$ is a spectral measure for the Hilbert space $\mathcal{H}$.
(b) For each $\mathcal{B}_2$-measurable function $\phi$ on $\Omega_2$ we have

$$\int_{\Omega_1} \phi(T(\omega_1)) E_1(d\omega_1) = \int_{\Omega_2} \phi(\omega_2) E_2(d\omega_2).$$

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Several times in the proof we will make use of a lemma on transformation of scalar-valued measures. Because of great similarity between this lemma and our theorem and for ease of reference, the lemma is stated below. (cf. [3, Theorem C, p. 163]).

3. Lemma. If $T$ is a measurable transformation from a measure space $(X, \mathcal{S}, \mu)$ into a measurable space $(Y, \mathcal{G})$, and if $g$ is an extended real-valued measurable function on $Y$, then

$$\int_Y g(d\mu T^{-1}) = \int_X (gT)d\mu,$$

in the sense that if either integral exists, then so does the other and the two are equal.

Proof of Theorem 2. (a) It is obvious that for each $B \in \mathcal{B}_2$, $E_2(B)$ is a projection operator on $\mathcal{F}$ and $E_2(\Omega_2) = I$. If $\{B_k\}$ is a disjoint sequence of sets in $\mathcal{B}_2$, then $\{T^{-1}(B_k)\}$ is a sequence of disjoint sets in $\mathcal{B}_1$, and hence

$$E_2\left(\bigcup_k B_k\right) = E_1\left(T^{-1}\left(\bigcup_k B_k\right)\right) = E_1\left(\bigcup_k T^{-1}(B_k)\right) = \sum_k E_1(T^{-1}(B_k)) = \sum_k E_2(B_k).$$

(b) We give the proof of (b) in two steps.

(Step 1). For a simple function $f = \sum_{k=1}^n a_k \chi_{B_k}$, where $\chi_{B_k}$ is the indicator function of $B_k \in \mathcal{B}_2$,

$$\int_{\Omega_2} f(\omega_2) E_2(d\omega_2) = \sum_{k=1}^n a_k E_2(B_k)$$

$$= \sum_{k=1}^n a_k E_1(T^{-1}(B_k))$$

$$= \int_{\Omega_1} \left(\sum_{k=1}^n a_k \chi_{T^{-1}(B_k)}\right)(\omega_1) E_1(d\omega_1)$$

$$= \int_{\Omega_1} \phi(T(\omega_1)) E_1(d\omega_1).$$

(Step 2). Let $\phi$ be any $\mathcal{B}_2$-measurable function on $\Omega_2$, and let

$$T_1 = \int_{\Omega_1} \phi(T(\omega_1)) E_1(d\omega_1) \& T_2 = \int_{\Omega_2} \phi(\omega_2) E_2(d\omega_2).$$
We first show that \( T_1 \) and \( T_2 \) have the same domain, i.e., \( \mathcal{D}_{T_1} = \mathcal{D}_{T_2} \).

For each \( x \) in \( \mathcal{K} \), by Lemma 3,

\[
\int_{\Omega_1} |\phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 = \int_{\Omega_2} |\phi(\omega_2)|^2 |E_2(d\omega_2)x|^2.
\]

Now

\[
x \in \mathcal{D}_{T_2} \iff \int_{\Omega_2} |\phi(\omega_2)|^2 |E_2(d\omega_2)x|^2 < \infty \quad \text{by (1)}
\]

\[
\iff \int_{\Omega_1} |\phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 < \infty \quad \text{by (2)}
\]

\[
\iff x \in \mathcal{D}_{T_1} \quad \text{by (1)}.
\]

Hence \( \mathcal{D}_{T_1} = \mathcal{D}_{T_2} \).

If \( x \in \mathcal{D}_{T_2} \), then by (1) and (3), \( \int_{\Omega_2} |\phi(\omega_2)|^2 |E_2(d\omega_2)x|^2 \) is equal to \( \int_{\Omega_1} |\phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 < \infty \). Therefore there exists a sequence of simple functions \( \{\phi_n(\omega_2)\}_{n=1}^{\infty} \) on \( \Omega_2 \) such that

\[
\lim_{n \to \infty} \int_{\Omega_2} |\phi_n(\omega_2) - \phi(\omega_2)|^2 |E_2(d\omega_2)x|^2 = 0.
\]

It easily follows, by (3) and (4), that the sequence of simple functions \( \{\phi_n(T(\omega_1))\}_{n=1}^{\infty} \) on \( \Omega_1 \) satisfies the relation

\[
\lim_{n \to \infty} \int_{\Omega_1} |\phi_n(T(\omega_1)) - \phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 = 0.
\]

From (2), (4) and (5) it follows that

\[
\left\{ \int_{\Omega_2} \phi_n(\omega_2)E_2(d\omega_2) \right\} x \to T_2x, \quad \text{as } n \to \infty,
\]

\[
\left\{ \int_{\Omega_1} \phi_n(T(\omega_1))E_2(d\omega_1) \right\} x \to T_1x, \quad \text{as } n \to \infty.
\]

Now

\[
T_2x = \lim_{n \to \infty} \left\{ \int_{\Omega_2} \phi_n(\omega_2)E_2(d\omega_2) \right\} x \quad \text{by (6)}
\]

\[
= \lim_{n \to \infty} \left\{ \int_{\Omega_1} \phi_n(T(\omega_1))E_1(d\omega_1) \right\} x \quad \text{by (Step 1)}
\]

\[
= T_1x \quad \text{by (6)}.
\]

Hence \( T_1 = T_2 \). Q.E.D.
References


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THE MOMENTS OF RECURRENCE TIME

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In connection with Poincaré's recurrence theorem Kac [1] obtained the mean of the recurrence time (formula (3) below) and the author [2] gave a very simple proof of this result. Recently Blum and Rosenblatt [3] obtained the higher moments (formula (2) below). In the present note we obtain both results by an exceedingly simple and perspicuous argument. This note is entirely self-contained.

Let \( \Omega \) be a point set, \( m \) a probability measure on \( \Omega \), and \( T \) a one-to-one ergodic measure-preserving transformation of \( \Omega \) into itself. Let \( A \subseteq \Omega \) be such that \( m(A) > 0 \). For any point \( a \) in \( \Omega \) let \( n(a) \) be the smallest positive integer such that \( T^n a \in A \); if no such integer exists let \( n(a) = \infty \). Define \( A_k = \{ a \in A \mid n(a) = k \} \), \( \overline{A} = \Omega - A \), and \( \Gamma_k = \{ a \in \overline{A} \mid n(a) = k \} \). Borrowing the notation of [3] we will define

\[
(1) \quad p_n = m\{ \Gamma_n \cup \Gamma_{n+1} \cup \cdots \},
\]

for \( n \geq 1 \). We will also make use of the usual combinatorial symbol \( (k)_{j} = k(k-1) \cdots (k-j+1) \) for \( k \) and \( j \) positive integers, with \( (k)_{0} = 1 \).

Our object will be to prove that

\[
(2) \quad D_j = \int_{A} [n(a)]_j dm = j(j - 1) \sum_{k=j-2}^{\infty} (k)_{j-2} p_{k+1}
\]

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2 These moments have also been obtained by F. H. Simons, Notice #40 of the Eindhoven Technical School, December 23, 1966.