A TRANSFORMATION THEOREM ON SPECTRAL MEASURES[1]

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1. Introduction. Let $\mathcal{B}$ be a σ-algebra of subsets of a given space Ω, and let $\boldsymbol{E}$ be a countably additive projection-valued measure for a Hilbert space $\mathcal{H}$, i.e., for each $B\in\mathcal{B}$, $\boldsymbol{E}(B)$ is a projection operator on $\mathcal{H}$ such that $\boldsymbol{E}(\bigcup_k B_k)=\sum_k \boldsymbol{E}(B_k)$ whenever $\{B_k\}$ is a sequence of disjoint sets in $\mathcal{B}$ and $\boldsymbol{E}(\Omega)=\boldsymbol{I}$, where $\boldsymbol{I}$ denotes the identity operator on $\mathcal{H}$. We will refer to $(\Omega, \mathcal{B}, \mathcal{H}, \boldsymbol{E})$ as a *spectral measure for the Hilbert space* $\mathcal{H}$.

For each element x in $\mathcal{H}$ the function $|\boldsymbol{E}(\cdot)x|^2$ is a countably additive nonnegative measure on $\mathcal{B}$ such that $|\boldsymbol{E}(\Omega)x|^2=|x|^2<\infty$. If ϕ is a $\mathcal{B}$-measurable function defined on Ω, then the integral $\int_\Omega\phi(\omega)\boldsymbol{E}(d\omega)$ is a well-defined linear operator $\boldsymbol{T}$ (not necessarily bounded) in $\mathcal{H}$ with domain

$$\mathfrak{D}_T = \left\{x: x\in\mathcal{H}\ \&\ \int_\Omega |\phi(\omega)|^2 |\boldsymbol{E}(d\omega)x|^2 < \infty\right\}. \tag{1}$$

It is a simple result that if $x\in\mathfrak{D}_T$, then

$$|Tx|^2 = \int_\Omega |\phi(\omega)|^2 |\boldsymbol{E}(d\omega)x|^2. \tag{2}$$

In this paper we propose to establish the following theorem concerning transformation of spectral measures for a Hilbert space $\mathcal{H}$.

2. Theorem. *Let*

(i) $(\Omega_1, \mathcal{B}_1, \mathcal{H}, \boldsymbol{E}_1)$ *be a spectral measure for a Hilbert space* $\mathcal{H}$,

(ii) T *be a single-valued transformation on* Ω_1 *into a space* Ω_2,

(iii) $\mathcal{B}_2=\{B: B\subseteq\Omega_2$ *and* $T^{-1}(B)\in\mathcal{B}_1\}$,

(iv) *for each* $B\in\mathcal{B}_2$,

$$\boldsymbol{E}_2(B) = \boldsymbol{E}_1(T^{-1}(B)).$$

Then (a) $(\Omega_2, \mathcal{B}_2, \mathcal{H}, \boldsymbol{E}_2)$ *is a spectral measure for the Hilbert space* $\mathcal{H}$.
(b) *For each* $\mathcal{B}_2$*-measurable function* ϕ *on* Ω_2 *we have*

$$\int_{\Omega_1}\phi(T(\omega_1))\boldsymbol{E}_1(d\omega_1) = \int_{\Omega_2}\phi(\omega_2)\boldsymbol{E}_2(d\omega_2).$$

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Several times in the proof we will make use of a lemma on transformation of scalar-valued measures. Because of great similarity between this lemma and our theorem and for ease of reference, the lemma is stated below. (cf. [**3**, Theorem C, p. 163]).

3. **Lemma.** *If T is a measurable transformation from a measure space $(\boldsymbol{X}, \mathcal{S}, \mu)$ into a measurable space $(\boldsymbol{Y}, \mathfrak{I})$, and if g is an extended real-valued measurable function on $\boldsymbol{Y}$, then*

$$\int_Y g d(\mu T^{-1}) = \int_X (gT) d\mu,$$

in the sense that if either integral exists, then so does the other and the two are equal.

Proof of Theorem 2. (a) It is obvious that for each $B \in \mathcal{B}_2$, $\boldsymbol{E}_2(B)$ is a projection operator on $\mathcal{H}$ and $\boldsymbol{E}_2(\Omega_2) = \boldsymbol{I}$. If $\{B_k\}$ is a disjoint sequence of sets in $\mathcal{B}_2$, then $\{T^{-1}(B_k)\}$ is a sequence of disjoint sets in $\mathcal{B}_1$, and hence

$$\begin{aligned} \boldsymbol{E}_2\left(\bigcup_k B_k\right) &= \boldsymbol{E}_1\left(T^{-1}\left(\bigcup_k B_k\right)\right) = \boldsymbol{E}_1\left(\bigcup_k T^{-1}(B_k)\right) \\ &= \sum_k \boldsymbol{E}_1(T^{-1}(B_k)) = \sum_k \boldsymbol{E}_2(B_k). \end{aligned}$$

(b) We give the proof of (b) in two steps.

(*Step* 1). For a simple function $\phi = \sum_{k=1}^n a_k \chi_{B_k}$, where χ_{B_k} is the indicator function of $B_k \in \mathcal{B}_2$,

$$\begin{aligned} \int_{\Omega_2} \phi(\omega_2) \boldsymbol{E}_2(d\omega_2) &= \int_{\Omega_2} \left(\sum_{k=1}^n a_k \chi_{B_k}\right)(\omega_2) \boldsymbol{E}_2(d\omega_2) = \sum_{k=1}^n a_k \boldsymbol{E}_2(B_k) \\ &= \sum_{k=1}^n a_k \boldsymbol{E}_1(T^{-1}(B_k)) \\ &= \int_{\Omega_1} \left(\sum_{k=1}^n a_k \chi_{T^{-1}(B_k)}\right)(\omega_1) \boldsymbol{E}_1(d\omega_1) \\ &= \int_{\Omega_1} \phi(T(\omega_1)) \boldsymbol{E}_1(d\omega_1). \end{aligned}$$

(*Step* 2). Let ϕ be any $\mathcal{B}_2$-measurable function on Ω_2, and let

$$\boldsymbol{T}_1 = \int_{\Omega_1} \phi(T(\omega_1)) \boldsymbol{E}_1(d\omega_1) \ \& \ \boldsymbol{T}_2 = \int_{\Omega_2} \phi(\omega_2) \boldsymbol{E}_2(d\omega_2).$$

We first show that T_1 and T_2 have the same domain, i.e., $\mathfrak{D}_{T_1}=\mathfrak{D}_{T_2}$. For each x in $\mathcal{H}$, by Lemma 3,

$$(3)\qquad \int_{\Omega_1} |\phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 = \int_{\Omega_2} |\phi(\omega_2)|^2 |E_2(d\omega_2)x|^2.$$

Now

$$\begin{aligned} x \in \mathfrak{D}_{T_2} &\Leftrightarrow \int_{\Omega_2} |\phi(\omega_2)|^2 |E_2(d\omega_2)x|^2 < \infty \qquad \text{by (1)} \\ &\Leftrightarrow \int_{\Omega_1} |\phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 < \infty \qquad \text{by (2)} \\ &\Leftrightarrow x \in \mathfrak{D}_{T_1} \qquad \text{by (1).} \end{aligned}$$

Hence $\mathfrak{D}_{T_1}=\mathfrak{D}_{T_2}$.

If $x \in \mathfrak{D}_{T_2}$, then by (1) and (3), $\int_{\Omega_2}|\phi(\omega_2)|^2|E_2(d\omega_2)x|^2$ is equal to $\int_{\Omega_1}|\phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2<\infty$. Therefore there exists a sequence of simple functions $\{\phi_n(\omega_2)\}_{n=1}^{\infty}$ on Ω_2 such that

$$(4)\qquad \lim_{n\to\infty} \int_{\Omega_2} |\phi_n(\omega_2) - \phi(\omega_2)|^2 |E_2(d\omega_2)x|^2 = 0.$$

It easily follows, by (3) and (4), that the sequence of simple functions $\{\phi_n(T(\omega_1))\}_{n=1}^{\infty}$ on Ω_1 satisfies the relation

$$(5)\qquad \lim_{n\to\infty} \int_{\Omega_1} |\phi_n(T(\omega_1)) - \phi(T(\omega_1))|^2 |E_1(d\omega_1)x|^2 = 0.$$

From (2), (4) and (5) it follows that

$$(6)\qquad \begin{aligned} \left\{\int_{\Omega_2} \phi_n(\omega_2)E_2(d\omega_2)\right\} x &\rightarrow T_2x, \quad \text{as } n\rightarrow\infty, \\ \left\{\int_{\Omega_1} \phi_n(T(\omega_1))E(d\omega_1)\right\} x &\rightarrow T_1x, \quad \text{as } n\rightarrow\infty. \end{aligned}$$

Now

$$\begin{aligned} T_2x &= \lim_{n\to\infty} \left\{\int_{\Omega_2} \phi_n(\omega_2)E_2(d\omega_2)\right\} x \qquad \text{by (6)} \\ &= \lim_{n\to\infty} \left\{\int_{\Omega 1} \phi_n(T(\omega_1))E_1(d\omega_1)\right\} x \qquad \text{by (Step 1)} \\ &= T_1x \qquad \text{by (6).} \end{aligned}$$

Hence $T_1=T_2$. Q.E.D.

References

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THE MOMENTS OF RECURRENCE TIME

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In connection with Poincaré's recurrence theorem Kac [1] obtained the mean of the recurrence time (formula (3) below) and the author [2] gave a very simple proof of this result. Recently Blum and Rosenblatt [3] obtained[2] the higher moments (formula (2) below). In the present note we obtain both results by an exceedingly simple and perspicuous argument. This note is entirely self-contained.

Let Ω be a point set, m a probability measure on Ω, and T a one-to-one ergodic measure-preserving transformation of Ω into itself. Let $A \subset \Omega$ be such that $m(A) > 0$. For any point a in Ω let $n(a)$ be the smallest positive integer such that $T^n a \in A$; if no such integer exists let $n(a) = \infty$. Define $A_k = \{a \in A \mid n(a) = k\}$, $\bar{A} = \Omega - A$, and $\Gamma_k = \{a \in \bar{A} \mid n(a) = k\}$. Borrowing the notation of [3] we will define

$$p_n = m\{\Gamma_n \cup \Gamma_{n+1} \cup \cdots\}, \tag{1}$$

for $n \geqq 1$. We will also make use of the usual combinatorial symbol $(k)_j = k(k-1) \cdots (k-j+1)$ for k and j positive integers, with $(k)_0 = 1$.

Our object will be to prove that

$$D_j = \int_A [n(a)]_j dm = j(j-1) \sum_{k=j-2}^{\infty} (k)_{(j-2)} p_{k+1} \tag{2}$$

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[2] These moments have also been obtained by F. H. Simons, Notice #40 of the Eindhoven Technical School, December 23, 1966.