AN EXAMPLE IN DIMENSION THEORY
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An example is constructed which justifies the following theorem:

**Theorem 2.** There exists a set $K \subseteq E^n$ such that, for each positive integer, $s, \dim K = \dim K^s = \dim K^a = n - 1$. Here $\dim$ means (inductive) topological dimension [1, p. 24], $K^s$ is the $s$-fold product of $K$ with itself and $K^a$ is the denumerable product of $K$ with itself.

This result is in contrast to the known result that if $A, B$ are nonvoid separable metric spaces with $A$ compact and $\dim B > 0$ then $\dim(A \times B) \geq \dim A$, equality holding only if $\dim A = \infty$.

The construction, for arbitrary $n$, is an exercise in the elementary geometry of $E^n$ and transfinite induction. However, for $n = 1$, a Cantor set or the rationals on the line will serve as an example; for $n = 2$, if the requirement that $K \subseteq E^n$ is deleted (or relaxed to $K \subseteq E^{n+1}$), the rationals in Hilbert space are an example. For $n > 2$, the "standard examples" [1, pp. 29, 64] of $n$-dimensional spaces contain cells, so that their arbitrary finite products exhibit increase in dimension.

I. Preliminaries. Let $\omega$ denote the set of positive integers. By continuum we mean a compact, connected, metric space. If $A$ is a set, $#(A)$ denotes the cardinality of $A$. Let $d$ be a minimal well-ordering of the unit interval, $I$; i.e., for $a \in d$, $c_a = \{ \beta \in d \mid \beta < \alpha \}$ has the property $#(c_a) < #(I) = c$.

A hyperplane of (linear) dimension $s$ in $E^r$ (the solution set of $(r - s)$ linearly independent linear equations) is denoted by $H^s$ or by $H$. It is well known that the topological dimension, $\dim H^s$, is $s$. Let $H$ denote the set of all linear translates of $H$. For $i = 1, 2$, let $H_i$ be a hyperplane of dimension $t_i$ in $E^r$. $H_1$ and $H_2$ are said to be in general position (with respect to each other) if for $H_i' \in \tilde{H}_i$, $H_i' \cap H_2' \neq \emptyset$ implies $H_i' \cap H_2' = H_t'$ where $t = \max[0, t_1 + t_2 - r]$. If $H_1$ and $H_2$ are in general position, we say that $H_1$ and $\tilde{H}_2$, and that $\tilde{H}_1$ and $H_2$, are in general position.

It is convenient to consider $(E^n)^s = E^{ns}$ as all "$s$-letter words," each letter being a point of $E^n$. Let $\tau_A : (E^n)^s \rightarrow (E^n)^t$ be defined by deleting...
the jth letter of \( w \in (E^n)^s \) for all \( j \in A \), where \( A \) is a proper subset of \( \omega_s = \{ j \in \omega \mid 1 \leq j \leq s \} \). Then \( t = s - \#(A) \). We abbreviate \( \tau_{[j]} \) as \( \tau_j \). Let \( \theta \) denote the origin in \( E^n \). For \( \emptyset \neq A \subset \omega_s \), define \( H(A) \) as the \( n \)-dimensional hyperplane which is the solution set of the equations \( \tau_j(p) = \tau_k(p) \) for \( j, k \in A \) and \( \tau_i(p) = \theta \) for \( p \in \omega_s \setminus A \). Let \( \gamma = \{ H(A) \mid \emptyset \neq A \subset \omega_s \} \). [Each \( H(A) \) may be thought of as a direction for \( n \)-dimensional hyperplanes. In this sense, if \( A = \omega_s \), \( H(A) \) represents the "diagonal" direction; if \( A = \{ j \} \), \( H(A) \) represents the direction parallel to the jth coordinate plane.]

The set \( S^k \) is a \( k \)-sphere in \( E^n \) iff \( S^k = S \cap H^{k+1} \) where \( \#(S^k) > 1 \) and \( S \) is an \( (r - 1) \) sphere; i.e., \( S \) is the set of all points at fixed positive distance from a given point of \( E^n \). Hence, \( S^k \) is the set of all points in \( H^{k+1} \) at fixed distance from a given point of \( H^{k+1} \).

Let \( M^k \) denote a \( k \)-dimensional Cantor-manifold [1, p. 93].

II. Lemmas. With reference to the notation of Theorem 2, \( K \) will be constructed so that the following Lemma 1 is applicable. Thus \( \dim K \geq n - 1 \) and therefore \( \dim K \geq n - 1 \). Lemma 1 is probably known, at least in the folklore.

**Lemma 1.** Let \( K \subset E^n \) such that for each nondegenerate continuum \( C \subset E^n \), \( K \cap C \neq \emptyset \). Then \( \dim K \geq n - 1 \).

**Proof.** Choose \( p_0 \in K \) and any \( U^\text{open} \subset E^n \ni p_0 \subset U_0 = U \) and \( \text{diam } U < 1 \). \( \overline{U} \setminus U \) separates \( E^n \), hence it contains an \( (n - 1) \)-dimensional Cantor-manifold, \( M^{n-1} \); \( K \cap (\overline{U} \setminus U) \supset K \cap M^{n-1} \neq \emptyset \) by hypothesis. Hence, \( \dim(K \cap M^{n-1}) \geq 0 \). If \( \dim(K \cap M^{n-1}) \geq s \) for all \( M^{n-1} \subset E^n \) then \( \dim K \geq s + 1 \). Inductively, let \( p_i \in M^{n-i} \subset E^n \) and let \( U_{n-i} \) be an open set in \( M^{n-i} \supset \overline{U}_{n-i} \supset M^{n-i} \). Then \( \overline{U}_{n-i} \setminus U_{n-i} \) separates \( M^{n-i} \) and therefore contains an \( (n - i - 1) \)-dimensional Cantor-manifold \( M^{n-i-1} \), \( i < n \). If \( \dim(K \cap M^{n-i-1}) \geq s \) for all such \( M^{n-i-1} \subset E^n \) then \( \dim(K \cap M^{n-i}) > s + 1 \). But for each \( M^1 \subset E^n \), \( K \cap M^1 \neq \emptyset \). Therefore \( \dim(K \cap M^1) \geq 0 \Rightarrow \dim(K \cap M^2) \geq 1 \Rightarrow \cdots \dim(K \cap M^{n-1}) \geq n - 2 \Rightarrow \dim K \geq n - 1 \).

Of course, \( \dim K = n \) iff \( K \) contains a nonnull open subset of \( E^n \).

We shall use, without explicit proof here, a weakened form of the following lemma, which asserts that hyperplanes may be tilted a small amount so that they are moved into general position with respect to a countable set of hyperplanes and continue to separate spheres about as they did before tilting.

**Lemma 2.** Given a countable collection of families of hyperplanes, \( \{ H_i \} \), a \( k \)-sphere \( S \), and a hyperplane \( H \), all in \( E^n \), such that \( S \setminus H = U_1 \cup U_2 \) where \( p \in U_1 \), \( U_1 \) is open and closed in \( S \setminus H \), and \( U_1 \cap U_2 = \emptyset \).
Then there exists a hyperplane $H'$ such that (1) $\dim H' = k$, (2) for each $i \in \omega$, $H'$ is in general position with respect to $H_i$, and (3) $S \setminus H' = V_1 \cup V_2$ where $p \in V_1 \cup U_1$, $V_i$ is open and closed in $S \setminus H'$, and $V_1 \cap V_2 = \emptyset$.

We now introduce some notations for use in Lemma 3 below. Choose a countable dense set of points in $E^n$ and the $(ns-1)$-dimensional spheres $S_{ns-1}$ with rational radii about them. For each $S_{ns-1}$, choose a countable set of $(ns-1)$-dimensional hyperplanes $H_{ns-1}$ so that their complementary domains form a basis for the topology of $S_{ns-1}$ and so that, with $\gamma$ defined as in §1, each $H_{ns-1}$ is in general position with respect to the $\tilde{H}^n \subset \gamma$. Lemma 2 assures us that this is possible. On each of the countably many $S_{ns-1}$'s, we choose countably many $S_{ns-2}$'s by $S_{ns-2} = S_{ns-1} \cap H_{ns-1}$, for the $H_{ns-1}$'s chosen above.

Inductively, for each chosen $S_{ns-k} = S_{ns-k+1} \cap H_{ns-k+1}$, choose a countable set of hyperplanes, $H_{ns-k}$, whose complementary domains in $S_{ns-k}$ form a basis for the topology of $S_{ns-k}$ and such that $H_{ns-k}$ is in general position with respect to $H_{ns-k}$. In this way, countably many spheres, $\{S_i\}$ are chosen, $ns - n \leq t < ns$. Denote $S_i^{ns-n}$ by $S_i$.

**Lemma 3.** Let $T \subseteq E^n$ such that, for each $i$, $T \cap S_i = \emptyset$, the spheres being chosen as above. Then $\dim T \leq n - 1$.

**Proof.** The complementary domains of the $S_i^{ns-k}$ in the $S_i^{ns-k+1}$ form a topological basis for $S_i^{ns-k+1}$, by construction. Therefore, $T \cap S_i = \emptyset \Rightarrow \dim(T \cap S_i) = -1 \Rightarrow \dim(T \cap S_{ns-n+1}) \leq 0 \Rightarrow \dim(T \cap S_{ns-n+2}) \leq 1 \Rightarrow \cdots \Rightarrow \dim(T \cap S_{ns-1}) \leq n - 2 \Rightarrow T \subseteq n - 1$.

**Lemma 4.** Let $K \subseteq E^n \supseteq \dim K < t \forall s \in \omega$. Then $\dim K^s < t$.

**Proof.** By [2, p. 126], it suffices to show that there is a sequence, $\{s_n\}$, of open covers of $K^s$:

(i) $\{s_n\} \subset \{s\}$,

(ii) order $\{s_n\} \leq t$,

(iii) mesh $\{s_n\} < 1/2^{t-2}$.

We consider $K \subset (0, 1)^n \times [0, 1]^n \times I^n$, the Hilbert cube. If $p, q \in I^n$, $p = \{p_i\}$, $q = \{q_i\}$ then $d(p, q) = \sum (1/2^i) |p_i - q_i|$. Construct an open cover, $V_1$, of $K \subset (0, 1)^n \supset \dim V_1 \leq t$ and mesh $V_1 < 1/2^{t-2}$. This is possible, since $\dim K < t$. Inductively, assume we have an open cover $V_i$ of $K^i \subset E^n \supset \dim V_i \leq t$ and mesh $V_i < 1/2^{t-2}$. Let $V_i = \{v \times (0, 1)^n | v \in V_i\}$. $V_i$ is an open cover of $K^{i+1} \subset E^{n+i}$, since $\dim K^{i+1} < t$, $\emptyset V_{i+1} < V_i$, order $V_{i+1} \leq t$ and mesh $V_{i+1} < 1/2^{t+1}$. Let $\{s_n\} = \{v \times \Pi_{j=n+1}^{\infty} (0, 1) | v \in V_i\}$. Since $V_{i+1} < V_i$, we have $\{s_n\} \subset \{s\}$. Order $\{s_n\}$ = order $V_{i+1}$ and mesh $\{s_n\} \leq \text{mesh } V_{i+2} 2 \sum_{j=n+1}^{\infty} \frac{1}{2^j} < 1/2^{t+i+2} + 2^{-n+1} < 1/2^{t-2}$.

Therefore, $\dim K^s < t$. 

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III. Theorems.

Theorem 1. Let $n, s \in \omega$. There exists $K \subseteq E^n$ such that $\dim K = \dim K^s = n - 1$.

Proof. We construct $K$ by transfinite induction. Let $\{S_i\}_{i \in \omega}$, $ns - n \leq r < ns$, be constructed as for Lemma 3, for the collection $\gamma$, with $S_i \subseteq H_i$ where $\dim H_i = ns - n + 1$. Let $R = \bigcup S_i$. Let $\mathcal{C} = \{C_\alpha\}$, $\alpha \in d$, be a minimal well-ordering of the nondegenerate continua contained in $E^n$. Consider $H \subseteq \tilde{H} \subseteq \gamma$, $C \subseteq \mathcal{C}$, and $S_i \cap H \cap H_i$ is a line, by Lemma 2. Therefore $S_i \cap H = S_i \cap H \cap H_i$ is at most 2 points.

Let $w$ be a $k$-letter word in $E^{nk}$, $0 \leq k < s$, $x \in E^n$ and $w(x)$ the (finite) set of all $s$-letter words in $E^s$ for $w' \in w(x)$ $\exists A \subseteq \omega$ for which $\tau_A(w') = w$ and $\tau_j(w') = x$, $j \in \omega \setminus A$. Let $A(w, \alpha) = \{x \mid x \in C_\alpha \text{ and } \exists w' \in w(x) \cap R\}$. $A(w, \alpha)$ is a countable set, since $w$ determines a finite set of $H_j \subseteq \tilde{H} \subseteq \gamma$, $(H \cap S_i) \leq 2$, and $A(w, \alpha) \subseteq \bigcup_{j \in \omega}(H_j \cap S_i)$.

Let $L \subseteq E^n$ and $A(L, \alpha) = \{x \mid x \in A(w, \alpha), w \in L^k, 0 \leq k < s\}$. If $(L) < c$ then $(A(L, \alpha)) < c$.

Consider $C_1$, the first element of $\mathcal{C}$. Let $H$ be the diagonal $n$-hyperplane in $E^{ns}$. Since $H \cap R$ is countable, there exists $x_0 \in C_1 \supseteq (x_0)^s \subseteq R$. Let $K_1 = \{x_0\}$.

Suppose that, for $\beta < \alpha$, the sets $K_\beta$ have been defined so that $\beta_1 < \beta_2 \Rightarrow K_{\beta_1} \subseteq K_{\beta_2}$, $(K_\beta) < c$, $C_\beta \cap K_\beta \neq \emptyset$, and $K_\beta \cap R = \emptyset$. Let $K'_\alpha = \bigcup_{\beta < \alpha} K_\beta$. Then $(K'_\alpha) < c$ and $(K'_\alpha) \cap R = \emptyset$. If $K'_\alpha \cap C_\alpha \neq \emptyset$, let $K_{\alpha} = K'_\alpha$. Otherwise, choose (appropriately) a point $p_\alpha \in C_\alpha$ and let $K_{\alpha} = K'_\alpha \cup \{p_\alpha\}$. The inductive properties, except $K_\alpha \cap R = \emptyset$, are clearly realized. Since $(K'_\alpha) < c$, $(A(K'_\alpha, \alpha)) < c$. But $(C_\alpha) = c$. Therefore, $C_\alpha \setminus A(K'_\alpha, \alpha) \neq \emptyset$. Hence, choose $p_\alpha \in C_\alpha \setminus A(K'_\alpha, \alpha)$ and $K_{\alpha} = K'_\alpha \cup \{p_\alpha\}$ satisfies $K_{\alpha} \cap R = \emptyset$.

Thus, we construct $K_\alpha \forall \alpha$. Let $K = \bigcup K_\alpha$. Then $K^s \cap R = \emptyset$. By Lemma 3, $\dim K^s \leq n - 1$. Therefore $\dim K \leq n - 1$. But Lemma 1 applies to $K$, by construction. Hence $\dim K \geq n - 1$. This implies that $\dim K^s \geq n - 1$. Thus, $\dim K = n - 1$.

A slight variant of the procedure above allows us to prove the following.

Theorem 2. There exists a set $K \subseteq E^n$ such that for each positive integer, $s$, $\dim K = \dim K^s = \dim K^s = n - 1$.

Proof. For each $s$, we construct the spheres $S_i^{ns-n}$ in $E^{ns}$. In the inductive definition of $K$, as above, we assume in addition that $K_s^s \cap S_i^{ns-n} = \emptyset$ for each $s$ and $i$. Let $K'_s = \bigcup_{\beta < \alpha} K_\beta$. For $L \subseteq E^n$ and $s \in \omega$, let $A(L, \alpha, s)$ be the set constructed in $E^{ns}$, and denoted by $A(L, \alpha)$, in Theorem 1. We have $(A(K'_s, \alpha, s)) < c$. Hence
#(\cup_{s \in \omega} A(K'_a, \alpha, s)) < c$ and $\exists p_a \in C_a \setminus \cup A(K'_a, \alpha, s)$. Let $K_a = K'_a \cup \{p_a\}$. Then $K_a$ fulfills the inductive assumptions and $K = \cup K_a$ satisfies $\dim K^s = n - 1$, for each $s \in \omega$. Hence, by Lemma 4, $\dim K'^s < n$. But $K^o \supset K'$ where $K' \supset K$ and $\dim K \geq n - 1$. Thus $\dim X = n - 1$.

References


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IMMERSIONS INTO MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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1. Introduction. Let $M$ and $M'$ denote $C^\infty$ Riemannian manifolds, $K$ and $K'$ their respective sectional curvature functions, and $\psi : M \rightarrow M'$ an isometric immersion. A consequence of Theorem 2 of [5] is that if at any point $m \in M$, $K(\pi) < K(d\psi(\pi))$, where $\pi$ is some plane in $M_m$, (the tangent space to $M$ at $m$) then there are no $\psi$ that immerse $M^d$ in $M^{d+k}$ unless $k$ is greater than or equal to $d - 1$. By restricting $M$ to be compact and $M'$ to be complete and simply connected, O'Neill has shown in [3] that there are no isometric immersions of $M^d$ in $M^{d+k}$ when $K \leq K' \leq 0$ on $M$ unless $k$ is greater than or equal to $d$. Amaral (Theorem A of [1]) considered immersions of compact $M^d$ in $H^{d+1}(\overline{C})$, $(d+1)$-dimensional hyperbolic space of curvature $\overline{C}$, and by only assuming $K \leq 0$ proved that there are no isometric immersions of $M^d$ in $H^{d+1}(\overline{C})$. Using methods similar to those of [3] we prove a theorem which strengthens O'Neill's result in the case that $M$ is of constant negative curvature and includes Amaral's result.

2. Results.

Theorem. Let $M$ be a compact $d$-dimensional Riemannian manifold and let $M'$ be a complete simply connected Riemannian manifold of constant curvature $\overline{C} \leq 0$ and of dimension less than $2d$. If the sectional curvature

Received by the editors July 13, 1966.

1 Part of the research in this paper was done while the author was the recipient of an NSF Research Participation award in Mathematics at the University of Oklahoma.