AN EXAMPLE IN DIMENSION THEORY

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An example is constructed which justifies the following theorem:

THEOREM 2. *There exists a set* $K \subset E^n$ *such that, for each positive integer,* s, $\dim K = \dim K^s = \dim K^\omega = n-1$. *Here* dim *means* (*inductive*) *topological dimension* [**1**, p. 24], K^s *is the s-fold product of K with itself and* K^ω *is the denumerable product of K with itself.*

This result is in contrast to the known result that if A, B are nonvoid separable metric spaces with A compact and $\dim B > 0$ then $\dim(A \times B) \geqq \dim A$, equality holding only if $\dim A = \infty$.

The construction, for arbitrary n, is an exercise in the elementary geometry of E^n and transfinite induction. However, for $n=1$, a Cantor set or the rationals on the line will serve as an example; for $n=2$, if the requirement that $K \subset E^n$ is deleted (or relaxed to $K \subset E^{n+1}$), the rationals in Hilbert space are an example. For $n>2$, the "standard examples" [**1**, pp. 29, 64] of n-dimensional spaces contain cells, so that their arbitrary finite products exhibit increase in dimension.

I. **Preliminaries.** Let ω denote the set of positive integers. By *continuum* we mean a compact, connected, metric space. If A is a set, $\#(A)$ denotes the cardinality of A. Let d be a minimal well-ordering of the unit interval, I; i.e., for $\alpha \in d$, $c_\alpha = \{\beta \in d \mid \beta < \alpha\}$ has the property $\#(c_\alpha) < \#(I) = c$.

A hyperplane of (linear) dimension s in E^r (the solution set of $(r-s)$ linearly independent linear equations) is denoted by H^s or by H. It is well known that the topological dimension, $\dim H^s$, is s. Let $\tilde{H}$ denote the set of all linear translates of H. For $i=1, 2$, let H_i be a hyperplane of dimension t_i in E^r. H_1 and H_2 are said to be in general position (with respect to each other) if for $H_i' \in \tilde{H}_i$, $H_1' \cap H_2' \neq \emptyset$ implies $H_1' \cap H_2' = H^t$ where $t = \max[0, t_1+t_2-r]$. If H_1 and H_2 are in general position, we say that H_1 and $\tilde{H}_2$, and that $\tilde{H}_1$ and $\tilde{H}_2$, are in general position.

It is convenient to consider $(E^n)^s = E^{ns}$ as all "s-letter words," each letter being a point of E^n. Let $\tau_A : (E^n)^s \to (E^n)^t$ be defined by deleting

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the jth letter of $w\in(E^n)^s$ for all $j\in A$, where A is a proper subset of $\omega_s=\{j\in\omega \mid 1\leqq j\leqq s\}$. Then $t=s-\#(A)$. We abbreviate $\tau_{\{j\}}$ as τ_j. Let θ denote the origin in E^n. For $\emptyset\neq A\subset\omega_s$, define $H(A)$ as the n-dimensional hyperplane which is the solution set of the equations $\tau_j(p)=\tau_k(p)$ for $j,\ k\in A$ and $\tau_i(p)=\theta$ for $p\in\omega_s\backslash A$. Let $\gamma=\{\tilde{H}(A) \mid \emptyset\neq A\subset\omega_s\}$. [Each $\tilde{H}(A)$ may be thought of as a direction for n-dimensional hyperplanes. In this sense, if $A=\omega_s$, $\tilde{H}(A)$ represents the "diagonal" direction; if $A=\{j\}$, $\tilde{H}(A)$ represents the direction parallel to the jth coordinate plane.]

The set S^k is a k-sphere in E^r iff $S^k=S\cap H^{k+1}$ where $\#(S^k)>1$ and S is an $(r-1)$ sphere; i.e., S is the set of all points at fixed positive distance from a given point of E^r. Hence, S^k is the set of all points in H^{k+1} at fixed distance from a given point of H^{k+1}.

Let M^k denote a k-dimensional Cantor-manifold [**1**, p. 93].

II. **Lemmas.** With reference to the notation of Theorem 2, K will be constructed so that the following Lemma 1 is applicable. Thus $\dim K\geqq n-1$ and therefore $\dim K^s\geqq n-1$. Lemma 1 is probably known, at least in the folklore.

LEMMA 1. *Let $K\subset E^n$ such that for each nondegenerate continuum $C\subset E^n$, $K\cap C\neq\emptyset$. Then* $\dim K\geqq n-1$.

PROOF. Choose $p_0\in K$ and any $U^{\text{open}}\subset E^n\ni p_0\in U_0=U$ and $\operatorname{diam} U<1$. $\overline{U}\backslash U$ separates E_n, hence it contains an $(n-1)$-dimensional Cantor-manifold, M^{n-1}; $K\cap(\overline{U}\backslash U)\supset K\cap M^{n-1}\neq\emptyset$ by hypothesis. Hence, $\dim(K\cap M^{n-1})\geqq 0$. If $\dim(K\cap M^{n-1})\geqq s$ for all $M^{n-1}\subset E^n$ then $\dim K\geqq s+1$. Inductively, let $p_i\in M^{n-i}\subset E^n$ and let U_{n-i} be an open set in $M^{n-i}\ni\overline{U}_{n-i}\not\supset M^{n-i}$. Then $\overline{U}_{n-i}\backslash U_{n-i}$ separates M^{n-i} and therefore contains an $(n-i-1)$-dimensional Cantor-manifold M^{n-i-1}, $i<n$. If $\dim(K\cap M^{n-i-1})\geqq s$ for all such $M^{n-i-1}\subset E^n$ then $\dim(K\cap M^{n-i})>s+1$. But for each $M^1\subset E^n$, $K\cap M^1\neq\emptyset$. Therefore $\dim(K\cap M^1)\geqq 0\Rightarrow\dim(K\cap M^2)\geqq 1\Rightarrow\cdots\dim(K\cap M^{n-1})\geqq n-2\Rightarrow\dim K\geqq n-1$.

Of course, $\dim K=n$ iff K contains a nonnull open subset of E^n.

We shall use, without explicit proof here, a weakened form of the following lemma, which asserts that hyperplanes may be tilted a small amount so that they are moved into general position with respect to a countable set of hyperplanes and continue to separate spheres about as they did before tilting.

LEMMA 2. *Given a countable collection of families of hyperplanes, $\{\tilde{H}_i\}$, a k-sphere S, and a hyperplane H, all in E^r, such that $S\backslash H=U_1\cup U_2$ where $p\in U_1$, U_i is open and closed in $S\backslash H$, and $U_1\cap U_2=\emptyset$.*

Then there exists a hyperplane H' *such that* (1) $\dim H'=k$, (2) *for each* $i\in\omega$, H' *is in general position with respect to* $\tilde{H}_i$, *and* (3) $S\backslash H' = V_1\cup V_2$ *where* $p\in V_1\subset U_1$, V_i *is open and closed in* $S\backslash H'$, *and* $V_1\cap V_2=\varnothing$.

We now introduce some notations for use in Lemma 3 below. Choose a countable dense set of points in E^{ns} and the $(ns-1)$-dimensional spheres S^{ns-1} with rational radii about them. For each S^{ns-1}, choose a countable set of $(ns-1)$-dimensional hyperplanes H^{ns-1} so that their complementary domains form a basis for the topology of S^{ns-1} and so that, with γ defined as in §I, each H^{ns-1} is in general position with respect to the $\tilde{H}^n\in\gamma$. Lemma 2 assures us that this is possible. On each of the countably many S^{ns-1}'s, we choose countably many S^{ns-2}'s by $S^{ns-2}=S^{ns-1}\cap H^{ns-1}$, for the H^{ns-1}'s chosen above.

Inductively, for each chosen $S^{ns-k}=S^{ns-k+1}\cap H^{ns-k+1}$, choose a countable set of hyperplanes, H^{ns-k}, whose complementary domains in S^{ns-k} form a basis for the topology of S^{ns-k} and such that H^{ns-k} is in general position with respect to $\tilde{H}^n\in\gamma$. In this way, countably many spheres, $\{S_i^t\}$ are chosen, $ns-n\leqq t<ns$. Denote S_i^{ns-n} by S_i.

LEMMA 3. *Let* $T\subset E^{ns}$ *such that, for each* i, $T\cap S_i=\varnothing$, *the spheres being chosen as above. Then* $\dim T\leqq n-1$.

PROOF. The complementary domains of the S_i^{ns-k} in the S_i^{ns-k+1} form a topological basis for S^{ns-k+1}, by construction. Therefore, $T\cap S_i=\varnothing\Rightarrow\dim(T\cap S_i)=-1\Rightarrow\dim(T\cap S^{ns-n+1})\leqq 0\Rightarrow\dim(T\cap S^{ns-n+2})\leqq 1\Rightarrow\cdots\Rightarrow\dim(T\cap S^{ns-1})\leqq n-2\Rightarrow\dim T\leqq n-1$.

LEMMA 4. *Let* $K\subset E^n\ni\dim K^s<t\,\forall s\in\omega$. *Then* $\dim K^\omega<t$.

PROOF. By [2, p. 126], it suffices to show that there is a sequence, $\mathfrak{U}_i$, of open covers of $K^\omega\ni$:

(i) $\mathfrak{U}_{i+1}\prec\mathfrak{U}_i$,

(ii) order $\mathfrak{U}_i\leqq t$,

(iii) mesh $\mathfrak{U}_i<1/2^{i-2}$.

We consider $K\subset(0,1)^n\approx E^n$ and $K^\omega\subset[(0_1,1)^n]^\omega\subset I^\omega$, the Hilbert cube. If $p,q\in I^\omega$, $p=\{p_i\}$, $q=\{q_i\}$ then $d(p,q)=\sum(1/2^i)|p_i-q_i|$.

Construct an open cover, V_1, of $K\subset(0,1)^n\ni$order $V_1\leqq t$ and mesh $V_1<\frac{1}{2}$. This is possible, since $\dim K<t$. Inductively, assume we have an open cover V_i of $K^i\subset E^{ni}\ni$order $V_i\leqq t$ and mesh $V_i<1/2^i$. Let $V_i'=\{v\times(0,1)^n|v\in V_i\}$. V_i' is an open cover of $K^{i+1}\subset E^{n(i+1)}$. Since $\dim K^{i+1}<t$, $\exists V_{i+1}$, an open cover of K^{i+1}, $\ni V_{i+1}\prec V_i'$, order $V_{i+1}\leqq t$ and mesh $V_{i+1}<1/2^{i+1}$. Let $\mathfrak{U}_i=\{v\times\prod_{j=ni+1}^{\infty}(0,1)^j|v\in V_i\}$. Since $V_{i+1}\prec V_i'$ we have $\mathfrak{U}_{i+1}\prec\mathfrak{U}_i$. Order $\mathfrak{U}_{i+1}=$order V_{i+1} and mesh $\mathfrak{U}_i\leqq$mesh $V_i+2\sum_{j=ni+1}^{\infty}1/2^j<1/2^i+2/2^{ni}<1/2^{i-2}$.

Therefore, $\dim K^\omega<t$.

III. Theorems.

THEOREM 1. *Let* n, $s \in \omega$. *There exists* $K \subset E^n$ *such that* $\dim K = \dim K^s = n-1$.

PROOF. We construct K by transfinite induction. Let $\{S_i^r\}_{i\in\omega}$, $ns-n \leqq r < ns$, be constructed as for Lemma 3, for the collection γ, with $S_i \subset H_i$ where $\dim H_i = ns-n+1$. Let $R = \bigcup S_i$. Let $\mathfrak{C} = \{C_\alpha\}$, $\alpha \in d$, be a minimal well-ordering of the nondegenerate continua contained in E^n. Consider $H \in \tilde{H} \in \gamma$, $C \in \mathfrak{C}$, and $S_i \cdot H_i \cap H$ is a line, by Lemma 2. Therefore $S_i \cap H = S_i \cap H \cap H_i$ is at most 2 points.

Let w be a k-letter word in E^{nk}, $0 \leqq k < s$, $x \in E^n$ and $w(x)$ the (finite) set of all s-letter words in $E^{ns} \ni$ for $w' \in w(x)$ $\exists A \subset \omega_s$ for which $\tau_A(w') = w$ and $\tau_j(w') = x$, $j \in \omega_s \setminus A$. Let $A(w, \alpha) = \{x \mid x \in C_\alpha$ and $\exists w' \in w(x) \cap R\}$. $A(w, \alpha)$ is a countable set, since w determines a finite set of $H_j \in \tilde{H} \in \gamma$, $\#(H \cap S_i) \leqq 2$, and $A(w, \alpha) \subset \bigcup_{j,i} (H_j \cap S_i)$.

Let $L \subset E^n$ and $A(L, \alpha) = \{x \mid x \in A(w, \alpha),\ w \in L^k,\ 0 \leqq k < s\}$. If $\#(L) < c$ then $\#[A(L, \alpha)] < c$.

Consider C_1, the first element of $\mathfrak{C}$. Let H be the diagonal n-hyperplane in E^{ns}. Since $H \cap R$ is countable, there exists $x_0 \in C_1 \ni (x_0)^s \notin R$. Let $K_1 = \{x_0\}$.

Suppose that, for $\beta < \alpha$, the sets K_β have been defined so that $\beta_1 < \beta_2 \Rightarrow K_{\beta_1} \subset K_{\beta_2}$, $\#(K_\beta) < c$, $C_\beta \cap K_\beta \neq \emptyset$, and $K_\beta^s \cap R = \emptyset$. Let $K_\alpha' = \bigcup_{\beta<\alpha} K_\beta$. Then $\#(K_\alpha') < c$ and $(K_\alpha')^s \cap R = \emptyset$. If $K_\alpha' \cap C_\alpha \neq \emptyset$, let $K_\alpha = K_\alpha'$. Otherwise, choose (appropriately) a point $p_\alpha \in C_\alpha$ and let $K_\alpha = K_\alpha' \cup \{p_\alpha\}$. The inductive properties, except $K_\alpha^s \cap R = \emptyset$, are clearly realized. Since $\#(K_\alpha') < c$, $\#A(K_\alpha', \alpha) < c$. But $\#(C_\alpha) = c$. Therefore, $C_\alpha \setminus A(K_\alpha', \alpha) \neq \emptyset$. Hence, choose $p_\alpha \in C_\alpha \setminus A(K_\alpha', \alpha)$ and $K_\alpha = K_\alpha' \cup \{p_\alpha\}$ satisfies $K_\alpha^s \cap R = \emptyset$.

Thus, we construct $K_\alpha \forall \alpha$. Let $K = \bigcup K_\alpha$. Then $K^s \cap R = \emptyset$. By Lemma 3, $\dim K^s \leqq n-1$. Therefore $\dim K \leqq n-1$. But Lemma 1 applies to K, by construction. Hence $\dim K \geqq n-1$. This implies that $\dim K^s \geqq n-1$. Thus, $\dim K^s = n-1$.

A slight variant of the procedure above allows us to prove the following.

THEOREM 2. *There exists a set* $K \subset E^n$ *such that for each positive integer*, s, $\dim K = \dim K^s = \dim K^\omega = n-1$.

PROOF. For each s, we construct the spheres S_i^{ns-n} in E^{ns}. In the inductive definition of K, as above, we assume in addition that $K_\beta^s \cap S_i^{ns-n} = \emptyset$ for each s and i. Let $K_\alpha' = \bigcup_{\beta<\alpha} K_\beta$. For $L \subset E^n$ and $s \in \omega$, let $A(L, \alpha, s)$ be the set constructed in E^{ns}, and denoted by $A(L, \alpha)$, in Theorem 1. We have $\#(A(K_\alpha', \alpha, s)) < c$. Hence

$\#(\bigcup_{s\in\omega} A(K_\alpha', \alpha, s)) < c$ and $\exists p_\alpha \in C_\alpha \setminus \bigcup A(K_\alpha', \alpha, s)$. Let $K_\alpha = K_\alpha' \cup \{p_\alpha\}$. Then K_α fulfills the inductive assumptions and $K = \bigcup K_\alpha$ satisfies dim $K^s = n-1$, for each $s \in \omega$. Hence, by Lemma 4, dim $K^\omega < n$. But $K^\omega \supset K'$ where $K' \overset{T}{\approx} K$ and dim $K \geqq n-1$. Thus dim $K^\omega = n-1$.

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IMMERSIONS INTO MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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1. Introduction. Let M and $\overline{M}$ denote C^∞ Riemannian manifolds, K and $\overline{K}$ their respective sectional curvature functions, and $\psi: M \to \overline{M}$ an isometric immersion. A consequence of Theorem 2 of [5] is that if at any point $m \in M$, $K(\pi) < \overline{K}(d\psi(\pi))$, where π is some plane in M_m, (the tangent space to M at m) then there are no ψ that immerse M^d in $\overline{M}^{d+k}$ unless k is greater than or equal to $d-1$. By restricting M to be compact and $\overline{M}$ to be complete and simply connected, O'Neill has shown in [3] that there are no isometric immersions of M^d in $\overline{M}^{d+k}$ when $K \leqq \overline{K} \leqq 0$ on M unless k is greater than or equal to d. Amaral (Theorem A of [1]) considered immersions of compact M^d in $H^{d+1}(\overline{C})$, $(d+1)$-dimensional hyperbolic space of curvature $\overline{C}$, and by only assuming $K \leqq 0$ proved that there are no isometric immersions of M^d in $H^{d+1}(\overline{C})$. Using methods similar to those of [3] we prove a theorem which strengthens O'Neill's result in the case that $\overline{M}$ is of constant negative curvature and includes Amaral's result.

2. Results.

THEOREM. *Let M be a compact d-dimensional Riemannian manifold and let $\overline{M}$ be a complete simply connected Riemannian manifold of constant curvature $\overline{C} \leqq 0$ and of dimension less than 2d. If the sectional*

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