A NOTE ON FINITELY GENERATED GROUPS

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We here generalize a result of Murasugi for finitely presented groups ([2]; see Corollary 3 below) to arbitrary finitely generated groups.

We begin with a simple extension of a well-known result for abelian groups.

THEOREM 1. *Let $G=HL$, where H is a subgroup of G and L is a subgroup of the center of G. If G/H is a direct product, $G/H=T_1\times K$, where K is a free abelian group, then G is a direct product, $G=T\times C$, where $H\lhd T$, $T/H\cong T_1$, $C\lhd L$ and $C\cong K$.*

In particular, if T_1 is finite then the index of H in T is finite.

PROOF. Since L is in the center of G, H is normal in $HL=G$. Let T be the preimage of T_1 under the natural homomorphism $G\rightarrow G/H$. For each element k of a fixed basis of K choose an element $c\in L$ which maps onto k and let C be the group generated by these elements c of L. The group C is in the center of G, so $C\lhd G$ and if $x\in C\cap T$, then $x=c_1^{r_1}c_2^{r_2}\cdots c_n^{r_n}$, so x maps onto $k_1^{r_1}\cdots k_n^{r_n}\in T_1\cap K=1$. Since K is free abelian $r_1=r_2=\cdots=r_n=0$, and thus $C\cap T=1$. Hence G is the direct product $T\times C$. Finally, $H\subseteq T$, so $C\cap H=1$ and $C\cong C/C\cap H\cong CH/H=K$, completing the proof.

COROLLARY 1. *Let $G=HL$, where H is a subgroup of G and L is a subgroup of the center of G. If G/H is finitely generated, then G is a direct product $G=T\times C$, where $H\lhd T$, $C\lhd L$ and the index of H in T is finite.*

PROOF. The group G/H is the direct product of a finite group T_1 and a free abelian group K. The corollary now follows from the theorem.

COROLLARY 2. *Let H be a subgroup of the finitely generated group A and let L be an abelian subgroup of $C_A(H)$, the centralizer of H in A. If HL is of finite index in A, then H is finitely generated.*

PROOF. The group $G=HL$ is finitely generated, because it has finite index in the finitely generated group A [3, 8.4.33]. As a homomorphic image of G, G/H must be finitely generated, so by corollary 1, $G=T\times C$ and H has finite index in T. But T is a homomorphic image of G, so T is finitely generated and hence H is finitely generated also.

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In particular, note that if Z is the center of the finitely generated group A and HZ has finite index in A, then H is finitely generated.

COROLLARY 3. *Let A be a finitely generated group and let H be a normal subgroup of A such that A/H is infinite cyclic. If the centralizer of H in A is not contained in H, then H is finitely generated.*

PROOF. If $C_A(H)\not\subset H$, let $x\in C_A(H)$, $x\notin H$ and let $L=\langle x\rangle$. Then HL/H is a nontrivial subgroup of the integers A/H and has finite index, so HL has finite index in A. The result now follows from Corollary 2.

COROLLARY 4. *Let A, H and L satisfy the assumptions of Corollary* 2. *Then* (i) *A is residually finite if and only if H is residually finite and* (ii) *A satisfies the maximal condition for subgroups if and only if H satisfies the maximal condition.*

PROOF. In the proof of corollary 2 we noted that $G=HL=T\times C$, where $H\lhd T$. If H is residually finite [**3**], then so is T, because it is a finite extension of H. Then $T\times C$ is residually finite, because the direct sum of residually finite groups is residually finite. Now $[G: HL]$ is finite so there is a normal subgroup N of G having finite index in G and contained in HL. Hence G is residually finite because N is residually finite as a subgroup of HL and a finite extension of a residually finite group is residually finite.

If H satisfies the maximal condition, so does HL because HL/H satisfies the maximal condition.

As before, there is a normal subgroup N of G of finite index in G which must satisfy the maximal condition, because it is a subgroup of HL. Hence G also satisfies the maximal condition.

The other statements are clear.

A group G is called an FC group if and only if every element of G has only a finite number of conjugates.

COROLLARY 5. *Let A be a finitely generated group with center Z. If H is a periodic FC group such that HZ has finite index in A, then A is an FC group.*

In particular, $[A:Z]$ *and* $[A':1]$ *are finite.*

PROOF. (See [**3**, Chapter **15**] for the material on FC groups used here.)

By the remark after Corollary 2, H is finitely generated, and a periodic FC group is locally normal, so H is a finite group. Note that $HZ/Z\cong H/H\cap Z$ is finite, so $[A: Z]=[A: HZ][HZ: Z]$ is finite. Hence A is an FC group.

Let $\phi(G)$ denote the Frattini subgroup of G.

COROLLARY 6. *Let A be a finitely generated group with center Z. Then* (i) *A is abelian if and only if $A=\phi(A)Z$ and* (ii) *A is an FC group if and only if $\phi(A)Z$ has finite index in A and $\phi(A)$ has an abelian subgroup S of finite index.*

PROOF. If $A=\phi(A)Z$, then by Corollary 2, $\phi(A)$ is finitely generated. Since $\phi(A)$ is the set of nongenerators of A we have that $\phi(A)Z=Z$, so $A=Z$ is abelian. The converse is clear.

If $\phi(A)Z$ has finite index in A, then the abelian group SZ has finite index in A, so $\phi(A)$ is finite and nilpotent by a result of P. Hall [**1**, Lemma 10] and hence $[A:Z]$ is finite. Thus A is an FC group.

Conversely, if A is an FC group, then $[A:Z]$ is finite and by Hall's result $\phi(A)$ is finite and nilpotent, so $\phi(A)$ has an abelian subgroup of finite index.

ADDED IN PROOF. We note that the idea in Theorem 1 can be extended, so that theorem 1 is a special case of the following

THEOREM A. *Let $V(L)$ be the variety of groups satisfying the identical relations L.*

If $G=HB$, $B\in V(L)$, $H\lhd G$, and G/H maps homomorphically onto a free group K in the variety $V(L)$, then G splits over K, that is, $G=TC$, $T\lhd G$, $C\cong K$ and $T\cap C=1$.

PROOF. Since $K\in V(L)$, $K\cong F/N$, where F is an ordinary free group on the set $\{f_i \mid i\in I\}$ and N is a fully invariant subgroup of F determined by the laws L.

Let $k_i=f_iN$, $i\in I$ and let T be the kernel of the homomorphism θ from G onto K. We may assume that $k_i\neq 1$, $i\in I$. Choose elements c_i in B but not T such that c_i maps onto k_i, $i\in I$. Let C be the subgroup of G generated by the c_i. Let $w=\prod_{m=1}^{q} c_{i_m}^{e_m}$, $e_m=\pm 1$, $i_m\in I$ be an element $T\cap C$. Then $w\theta=\prod_{m=1}^{q} k_{i_m}^{e_m}=1$ in K, so $\prod_{m=1}^{q} x_{i_m}^{e_m}=1$ is an identical relation on any group of the variety $V(L)$. Now $C\subseteq B$, so $C\in V(L)$, and we have that $w=1$, so $T\cap C=1$. Finally, $K\cong G/T=CT/T\cong C/C\cap T\cong C$.

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3. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

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