A NOTE ON FINITELY GENERATED GROUPS

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We here generalize a result of Murasugi for finitely presented groups ([2]; see Corollary 3 below) to arbitrary finitely generated groups.

We begin with a simple extension of a well-known result for abelian groups.

**Theorem 1.** Let \( G = HL \), where \( H \) is a subgroup of \( G \) and \( L \) is a subgroup of the center of \( G \). If \( G/H \) is a direct product, \( G/H = T_1 \times K \), where \( K \) is a free abelian group, then \( G \) is a direct product, \( G = T \times C \), where \( H \triangleleft T_1 \), \( T/H \cong T_1 \), \( C \triangleleft L \) and \( C \cong K \).

In particular, if \( T_1 \) is finite then the index of \( H \) in \( T \) is finite.

**Proof.** Since \( L \) is in the center of \( G \), \( H \) is normal in \( HL = G \). Let \( T \) be the preimage of \( T_1 \) under the natural homomorphism \( G \rightarrow G/H \). For each element \( k \) of a fixed basis of \( K \) choose an element \( c \in L \) which maps onto \( k \) and let \( C \) be the group generated by these elements \( c \) of \( L \). The group \( C \) is in the center of \( G \), so \( C \triangleleft G \) and if \( x \in C \cap T \), then \( x = c_1^{r_1} c_2^{r_2} \cdots c_n^{r_n} \), so \( x \) maps onto \( k_1^{r_1} \cdots k_n^{r_n} \in T_1 \cap K = 1 \). Since \( K \) is free abelian \( r_1 = r_2 = \cdots = r_n = 0 \), and thus \( C \cap T = 1 \). Hence \( G \) is the direct product \( T \times C \). Finally, \( H \subseteq T \), so \( C \cap H = 1 \) and \( C \cong C/C \cap H \cong CH/H = K \), completing the proof.

**Corollary 1.** Let \( G = HL \), where \( H \) is a subgroup of \( G \) and \( L \) is a subgroup of the center of \( G \). If \( G/H \) is finitely generated, then \( G \) is a direct product \( G = T \times C \), where \( H \triangleleft T_1 \), \( C \triangleleft L \) and the index of \( H \) in \( T \) is finite.

**Proof.** The group \( G/H \) is the direct product of a finite group \( T_1 \) and a free abelian group \( K \). The corollary now follows from the theorem.

**Corollary 2.** Let \( H \) be a subgroup of the finitely generated group \( A \) and let \( L \) be an abelian subgroup of \( C_A(H) \), the centralizer of \( H \) in \( A \). If \( HL \) is of finite index in \( A \), then \( H \) is finitely generated.

**Proof.** The group \( G = HL \) is finitely generated, because it has finite index in the finitely generated group \( A \) [3, 8.4.33]. As a homomorphic image of \( G \), \( G/H \) must be finitely generated, so by corollary 1, \( G = T \times C \) and \( H \) has finite index in \( T \). But \( T \) is a homomorphic image of \( G \), so \( T \) is finitely generated and hence \( H \) is finitely generated also.

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In particular, note that if $Z$ is the center of the finitely generated group $A$ and $HZ$ has finite index in $A$, then $H$ is finitely generated.

**Corollary 3.** Let $A$ be a finitely generated group and let $H$ be a normal subgroup of $A$ such that $A/H$ is infinite cyclic. If the centralizer of $H$ in $A$ is not contained in $H$, then $H$ is finitely generated.

**Proof.** If $C_A(H) \subseteq H$, let $x \in C_A(H)$, $x \notin H$ and let $L = \langle x \rangle$. Then $HL/H$ is a nontrivial subgroup of the integers $A/H$ and has finite index, so $HL$ has finite index in $A$. The result now follows from Corollary 2.

**Corollary 4.** Let $A$, $H$ and $L$ satisfy the assumptions of Corollary 2. Then (i) $A$ is residually finite if and only if $H$ is residually finite and (ii) $A$ satisfies the maximal condition for subgroups if and only if $H$ satisfies the maximal condition.

**Proof.** In the proof of corollary 2 we noted that $G = HL = T \times C$, where $H \leq T$. If $H$ is residually finite [3], then so is $T$, because it is a finite extension of $H$. Then $T \times C$ is residually finite, because the direct sum of residually finite groups is residually finite. Now $[G:HL]$ is finite so there is a normal subgroup $N$ of $G$ having finite index in $G$ and contained in $HL$. Hence $G$ is residually finite because $N$ is residually finite as a subgroup of $HL$ and a finite extension of a residually finite group is residually finite.

If $H$ satisfies the maximal condition, so does $HL$ because $HL/H$ satisfies the maximal condition.

As before, there is a normal subgroup $N$ of $G$ of finite index in $G$ which must satisfy the maximal condition, because it is a subgroup of $HL$. Hence $G$ also satisfies the maximal condition.

The other statements are clear.

A group $G$ is called an FC group if and only if every element of $G$ has only a finite number of conjugates.

**Corollary 5.** Let $A$ be a finitely generated group with center $Z$. If $H$ is a periodic FC group such that $HZ$ has finite index in $A$, then $A$ is an FC group.

In particular, $[A:Z]$ and $[A':1]$ are finite.

**Proof.** (See [3, Chapter 15] for the material on FC groups used here.)

By the remark after Corollary 2, $H$ is finitely generated, and a periodic FC group is locally normal, so $H$ is a finite group. Note that $HZ/Z \cong H/H \cap Z$ is finite, so $[A:Z] = [A:HZ][HZ:Z]$ is finite. Hence $A$ is an FC group.
Let $\phi(G)$ denote the Frattini subgroup of $G$.

**Corollary 6.** Let $A$ be a finitely generated group with center $Z$. Then (i) $A$ is abelian if and only if $A = \phi(A)Z$ and (ii) $A$ is an FC group if and only if $\phi(A)Z$ has finite index in $A$ and $\phi(A)$ has an abelian subgroup $S$ of finite index.

**Proof.** If $A = \phi(A)Z$, then by Corollary 2, $\phi(A)$ is finitely generated. Since $\phi(A)$ is the set of nongenerators of $A$ we have that $\phi(A)Z = Z$, so $A = Z$ is abelian. The converse is clear.

If $\phi(A)Z$ has finite index in $A$, then the abelian group $SZ$ has finite index in $A$, so $\phi(A)$ is finite and nilpotent by a result of P. Hall [1, Lemma 10] and hence $[A: Z]$ is finite. Thus $A$ is an FC group.

Conversely, if $A$ is an FC group, then $[A: Z]$ is finite and by Hall's result $\phi(A)$ is finite and nilpotent, so $\phi(A)$ has an abelian subgroup of finite index.

**Added in Proof.** We note that the idea in Theorem 1 can be extended, so that theorem 1 is a special case of the following

**Theorem A.** Let $V(L)$ be the variety of groups satisfying the identical relations $L$.

If $G = HB$, $B \subseteq V(L)$, $H \triangleleft G$, and $G/H$ maps homomorphically onto a free group $K$ in the variety $V(L)$, then $G$ splits over $K$, that is, $G = TC$, $T \triangleleft G$, $C \cong K$ and $T \cap C = 1$.

**Proof.** Since $K \in V(L)$, $K \cong F/N$, where $F$ is an ordinary free group on the set $\{f_i | i \in I\}$ and $N$ is a fully invariant subgroup of $F$ determined by the laws $L$.

Let $k_i = f_i N$, $i \in I$ and let $T$ be the kernel of the homomorphism $\theta$ from $G$ onto $K$. We may assume that $k_i \neq 1$, $i \in I$. Choose elements $c_i$ in $B$ but not $T$ such that $c_i$ maps onto $k_i$, $i \in I$. Let $C$ be the subgroup of $G$ generated by the $c_i$. Let $w = \prod_{m=1}^{q} c_i^{e_m}$, $e_m = \pm 1$, $i_m \in I$ be an element $T \cap C$. Then $w \theta = \prod_{m=1}^{q} k_i^{e_m} = 1$ in $K$, so $\prod_{m=1}^{q} c_i^{e_m} = 1$ is an identical relation on any group of the variety $V(L)$. Now $C \subseteq B$, so $C \subseteq V(L)$, and we have that $w = 1$, so $T \cap C = 1$. Finally, $K \cong G/T = CT/T \cong C/C \cap T \cong C$.

**References**


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