ON SIMILAR BASES IN BARRELLED SPACES

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1. Introduction. By a *biorthogonal system* (x_α, f_α) in a linear topological space E we mean a family (x_α) of points of E (possibly uncountable) and a family (f_α) of points of E^* such that $f_\alpha(x_\beta) = \delta_{\alpha\beta}$, the Kronecker delta. If the family (f_α) is total over E, i.e., $f_\alpha(x) = 0$ for each α implies $x = 0$, then (x_α, f_α) is called [2] a *generalized basis* and if (x_n, f_n) is a biorthogonal sequence such that for each $x \in E$

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \tag{1.1}$$

then (x_n, f_n) is called a *Schauder basis* for E.

If E and F are linear topological spaces with biorthogonal systems (x_α, f_α) and (y_α, g_α) respectively, then (x_α) and (y_α) are *similar* provided that the collection of generalized sequences $(f_\alpha(x))$, $x \in E$, is precisely the collection of generalized sequences $(g_\alpha(y))$, $y \in F$. We assume, of course, that both families are indexed by the same family of indices.

It is obvious that if T is a linear homeomorphism from E onto F, (x_α, f_α) a biorthogonal system for E, and $T(x_\alpha) = y_\alpha$, then $(y_\alpha, f_\alpha T^{-1})$ is a biorthogonal system for F and (x_α) and (y_α) are similar.

The converse problem is of considerable interest. In [1] Arsove proved for E and F Fréchet spaces that Schauder bases (x_n, f_n) and (y_n, g_n) in E and F, respectively, are similar if and only if there is a linear homeomorphism T from E onto F such that $T(x_n) = y_n$ for each n. The proof is quite elaborate and the hypothesis that the spaces are Fréchet spaces is used to full advantage.

In [2] Arsove and Edwards were able to relax both the local convexity and the convergence requirements of (1.1) and thus obtain the isomorphism theorem (2.1 below) for generalized bases in complete metric linear spaces.

(However, there still remains some real merit in the discussion in [1], for the development there carries over to *absolutely similar* Schauder bases (treated in the second part of [1]), where the linear methods seems to shed no light at all.)

The purpose of this note is to give a concise proof of the isomorphism theorem for Schauder bases in a very general setting and to

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give some examples illustrating the relationships between our theorem and the theorems of [1] and [2].

2. **The isomorphism theorem.** A linear topological space has the *t-property* if each absorbing set is somewhere dense. It is easy to show that every linear topological space which is of the second category in itself has the t-property. A locally convex linear topological space is *barrelled* if each convex absorbing set is somewhere dense.

2.1. THEOREM. *Suppose that E and F are either*

(a) *linear topological spaces having the t-property, or,*

(b) *barrelled spaces.*

Suppose that (x_i, f_i) and (y_i, g_i) are Schauder bases for E and F respectively. Then (x_i) and (y_i) are similar if and only if there is a linear homeomorphism T from E onto F such that $T(x_n) = y_n$ for each n.

PROOF. The necessity presents the only difficulties. To prove the necessity, we represent arbitrary points x in E as

$$x = \sum_{n=1}^{\infty} f_n(x) x_n$$

and define

$$T_m(x) = \sum_{n=1}^{m} f_n(x) y_n \quad (m = 1, 2, 3, \cdots)$$

and

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n,$$

convergence being ensured by the assumed similarity property. Since each f_n is continuous the linear mappings T_m are all continuous. Moreover, it is clear that T is a one-to-one mapping of E onto F and that $T_m(x) \to T(x)$ for each $x \in E$. In particular the family $\{T_m\}$ is pointwise bounded on E and T is in the pointwise closure of $\{T_m\}$. By the Banach-Steinhaus Theorem [5, Theorem 5, p. 225] T is continuous. By symmetry the same is true of T^{-1} proving that T is indeed the desired isomorphism of E onto F.

It is interesting that neither metrizibility nor completeness enters into the proof.

3. **Examples.** Obviously our theorem includes the isomorphism theorem of [1] but not that of [2], since we maintain the convergence requirement.

Our first example shows that Theorem 2.1 is strictly stronger than that in [**1**] and that it applies to a class of spaces not included in [**2**].

3.1. EXAMPLE. Let X be a reflexive nonnormable Fréchet space with a Schauder basis (x_i, f_i) and let $E = X^*$ endowed with the strong topology. Then E is barrelled, nonmetrizible and (f_i, Jx_i) (J the canonical map from $X \to X^{**}$) is a Schauder basis for E. Thus 2.1 applies. The above example can be realized by taking $X = H(D)$, the holomorphic functions on D: $|z| < 1$ with the compact-open topology or by taking $X = (s)$, the countable product of the real line with the usual product topology. (See [**4**, p. 284].)

Our next examples show that the isomorphism theorem is false if one of the spaces fails to be barrelled.

3.2. EXAMPLE. Let E be a Banach space with a Schauder basis (x_n, f_n), and let F be E endowed with the weak topology. Clearly (x_n) (in E) and (x_n) (in F) are similar but E and F are not even homeomorphic.

Our next example perhaps is more interesting since both spaces are normed linear spaces.

3.3. EXAMPLE. Let E be l^1 and let (x_n, f_n) denote the unit vector basis. Let F be l^1 viewed as a dense subspace of (c_0) and let (y_n, g_n) denote the unit vector basis of F. Clearly (x_n) and (y_n) are similar but E and F are not isomorphic.

Our final example shows that the isomorphism theorem fails for generalized bases even if both spaces are complete and barrelled.

3.4. EXAMPLE. Let E be an infinite dimensional Banach space with a generalized basis (x_α, f_α) and let F be E endowed with the strongest locally convex topology [**3**, p. 153]. Then E and F are complete barrelled spaces. Clearly (x_α, f_α) is a generalized basis in F and (x_α) (in E) and (x_α) (in F) are similar. However E and F are not homeomorphic, for F is not metrizible.

Since the isomorphism theorem is true for generalized bases in complete metric linear spaces, it seems reasonable that the full strength of the Schauder basis requirement is not needed in barrelled spaces. It remains an open question as to what type of conditions on generalized bases are needed in (complete) barrelled spaces to preserve the isomorphism theorem.

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THE HOMOGENEOUS HILBERT BOUNDARY PROBLEM IN A BANACH ALGEBRA. II

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In an earlier paper [1] we solved the Hilbert boundary problem on the supposition that

$$\operatorname{Ind}[\varphi(t, \lambda)]_L \equiv \frac{1}{2\pi i}\{\log[\varphi(t, \lambda)]\}_L = 0$$

(uniformly with respect to $\lambda \in \Delta$).

In the present paper, which is a continuation of the paper [1], we shall ask for the solution of the problem *supposing that*

$$\operatorname{Ind}[\varphi(t, \lambda)]_L = n$$

uniformly with respect to $\lambda \in \Delta$, where n is a fixed *positive* integer. This condition implies the multiple-valuedness of the function $\log[\varphi(t, x)]$ on L, for

$$\operatorname{Ind}[\varphi(t, x)]_L = e \operatorname{Ind}[\varphi(t, \lambda)]_L.$$

This relation follows directly from the hypothesis on the index of $\varphi(t, \lambda)$ on L and the formula (3) of [1] for $\log[\varphi(t, x)]$.

In the present situation, *the solution is given by*

$$\phi^+(\zeta) = P(\zeta) \exp [F^+(\zeta, x)],$$
$$\phi^-(\zeta) = \zeta^{-n}P(\zeta) \exp[F^-(\zeta, x)],$$

where $P(\zeta)$ is any polynomial of degree n with coefficients in $\mathfrak{B}_0$, and

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