SCHAUDER DECOMPOSITIONS IN \((m)\)

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1. Introduction. If the Banach space \(B\) is a countably infinite direct sum of nontrivial subspaces \(B_i\), \(B = \sum B_i\), such that each \(b\) has a unique representation \(b = \sum b_i\), \(b_i \in B_i\); and if each of the projections \(P_ib = b_i\) is continuous then say that \(B\) has a Schauder decomposition (a brief history of this topic is found in [8]). Sanders [9] has shown that \((m)\), the space of bounded sequences, does not have a Schauder decomposition such that the coordinate sequences \((\delta_{ij})_{i,j=1}^{\infty}\) lie in distinct \(B_k\). The purpose of this note is to prove \((m)\) has no Schauder decomposition whatsoever. In fact this is true for a class of spaces which include those \(C(H)\) spaces for which \(H\) is compact, Hausdorff, and extremely disconnected; (and all direct factors of such spaces (the \(P_\lambda\)-spaces) (see [1, pp. 94-96] for a discussion of such \(C(H)\)).

Let \(B^*\) be the conjugate space of \(B\) and let \((f_n)\) be a sequence in \(B^*\). If \(\lim_{n} f_n(b)\) exists and is finite for every \(b\) in \(B\) then the limit defines an element of \(B^*\) [2, p. 52]. The following properties of \((m)\) are critical for our proof.

(1) If \(\lim_{n} f_n(b) = f(b)\) exists for each \(b\) in \((m)\) then \(\lim_{n} F(f_n)\) exists for each \(F\) in \((m)^{**}\) and is equal to \(F(f)\), [3, pp. 168, 169]. That is, sequential weak* convergence in \((m)^*\) implies weak convergence.

(2) Let \(X, Y\) be Banach spaces and let \(S, T\) be weakly compact linear operators such that

\[
X \xrightarrow{(m)} T
\]

Then \(TS: X \to Y\) is compact [2, p. 494].

Replacing \((m)\) with \(B\) in (1) and (2) our main theorem is: If \(B\) has properties (1) and (2) then it does not have a Schauder decomposition.

2. The main theorem. Suppose now that \(B\) has property (1) and a Schauder decomposition \(\sum B_j\). Then \(b = \sum_{j=1}^{\infty} P_j(b)\) for each \(b\) in \(B\). Define conjugate projections \(Q_j\) in \(B^*\) by \(Q_j f(b) = f(P_j b)\) for every \(b\) in \(B\), \(f\) in \(B^*\). Each sum \(\sum_{j=1}^{\infty} P_j\) is a continuous projection and has conjugate projection \(\sum_{j=1}^{\infty} Q_j\). The set of projections \(\sum_{j=1}^{\infty} P_j\) is uniformly norm bounded (in \(m, n\)) since \((\sum_{j=1}^{\infty} P_j)(b) = \sum_{j=1}^{\infty} b_j \to 0\), \(n, m \to \infty\) [2, p. 52]. As \(\|\sum_{j=1}^{n} Q_j\| = \|\sum_{j=1}^{n} P_j\|\) the set of projections

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\[ \sum_{n=1}^{m} Q_j \] is uniformly norm bounded as well. Moreover for each \( f \) in \( B^* \) one has

\[ \left( \sum_{n} Q_j f \right)(b) = \sum_{n} f(P_j b) = f \left( \sum_{n} P_j b \right) \to 0, \quad n, m \to \infty. \]

Then, using (1), \( \sum_{1}^{\infty} Q_j f \) converges weakly to \( f \). It is easy to see that the sequence \( (Q_j f) \) is unique. Thus, in this sense, \( B^* \) has a weak decomposition \( B^* = \sum_{1}^{\infty} B_j' \), where \( B_j' = Q_j B^* \). In fact \( \sum B_j' \) is a Schauder decomposition of \( B^* \) ((i), (ii) below).

**Proposition.** If \( B \) has a Schauder decomposition and satisfies (1) then

(i) If \( b_j \in B_j \) for each \( j \) and either \( b_j = 0 \) or \( \|b_j\| = 1 \) then \( \{b_j | b_j \neq 0\} \) is a basis for \( [b_j] \) (the closed span of \( (b_j) \)).

(ii) If \( f_j \in B_j' \) for each \( j \) and either \( f_j = 0 \) or \( \|f_j\| = 1 \) then \( \{f_j | f_j \neq 0\} \) is a basis for \( [f_j] \).

(iii) \( \sum B_j' \) is a Schauder decomposition of \( B^* \).

(iv) \( [f_j] \) in (ii) is reflexive.

(v) Let \( F \subseteq B^{**} \) \( (= (B*)^* \) and let \( F_n \) be \( F \) restricted to \( \sum_{1}^{\infty} B_j' \). Then \( \|F_n\| \to 0 \) (the decomposition is shrinking in the sense of \( [8] \)).

(vi) \( [b_j] \) in (i) above is reflexive.

Parts (iv) and (vi) are those needed to prove the main theorem, the others being steps to achieving (iv) and (vi).

Suppose now the proposition is known and suppose \( B \) satisfies (1), (2), and has Schauder decomposition \( \sum B_j \). Choose \( b_j \in B_j \) with \( \|b_j\| = 1 \). Let \( B^* = \sum B_j' \) be the decomposition in (iii). Choose \( c \) such that \( \|Q_j\| \leq c \) for every \( j \). Select \( g_j \) having norm one such that \( \|g_j(b_j)\| \geq 1/2 \). Then \( \|Q_j g_j(b_j)\| \geq 1/2 \) while \( \|Q_j g_j\| \leq c \). Set \( f_j = Q_j g_j \|Q_j g_j\| \). Then \( 1 = \|f_j\| \geq \|f_j(b_j)\| \geq 1/2c > 0 \). Find such \( f_j \) in \( B_j' \) for each \( j \). Recall that if \( i \neq j \) then \( f_j(b_i) = 0 \) \( (Q_j f_j(P_j b_i) = f_j(P_j P_i b_i) = f_j(0)) \). Consider the mapping

\[ [b_j] \xrightarrow{I} B \xrightarrow{T} [f_j]^* \]

where \( I \) is the identity and \( T \) is defined by \( Tb(\sum t_j f_j) = \sum t_j f_j(b) \). Then \( I \) and \( T \) are continuous and linear. Since by (iv) and (vi) both \([b_j]\) and \([f_j]\) are reflexive \([f_j]^*\) is also, and so \( I \) and \( T \) are weakly compact, \([1, p. 56]\). Thus by (2) \( TI \) is compact. Now \( T Ib_j(f_j) = T b_j(f_j) = f_j(b_j) \) so \( \|TIb_j\| \geq 1/2c \). A subsequence \( T Ib_{n_j} \) converges to an element of \([f_j]^*\) while \( T Ib_{n_j} \) converges weakly to 0 since

\[ T Ib_{n_j} \left( \sum_{1}^{\infty} t_j f_j \right) = \sum_{1}^{\infty} t_j f_j(b_{n_j}) = t_{n_j} f_{n_j}(b_{n_j}) \jmath \to 0. \]
and in a reflexive space weak and weak* convergence are the same [1, p. 25]. Thus $Tb_{n,j}$ converges in norm to 0 which is a contradiction to $\|Tb_j\| \geq 1/2c$ for all $j$.

3. Proof of the proposition. We shall need two well known theorems about bases, the first due to James [4], and the second to Karlin [5] (see [1, p. 69] for a proof). A basis $(b_n)$ for $B$ is boundedly complete if $\|\sum_{i=1}^n t_i b_i\| \leq K$ for all $n$ implies $\sum_{i=1}^\infty t_i b_i$ converges. It is shrinking if, for each $f$ in $B^*$ $f_n = f|_{[b_j, j \geq n]}$ (f restricted to the closed span of $\{b_j, j \geq n\}$) converges to 0 ($\|f_n\| \to 0$).

(J) Let $B$ have basis $(b_n)$. Then $B$ is reflexive if and only if $(b_n)$ is shrinking and boundedly complete.

A sequence $(b_n)$ is a weak Schauder basis if for each $b$ in $B$ there is a unique sequence $(t_n)$ such that $\sum t_n b_n$ converges weakly to $b$ and if the functionals $(\phi_i(b)) = t_i$ are continuous.

(K) A weak Schauder basis is a basis.

Both (i) and (ii) will follow from (iii) and its proof. To show (iii), that is that $\sum B_j$ is a Schauder decomposition, we must show $f = \sum Q_j f$ for every $f$ in $B^*$. Now $\sum_{i=1}^\infty Q_i f$ converges weakly* to $f$ and so weakly to $f$. Thus each $f$ has a unique representation $(Q_i f)$ such that $\sum_{i=1}^\infty Q_i f$ converges weakly to $f$ and the coordinate projection $Q_i$ is continuous for each $j$. Such a decomposition we may call a weak decomposition. The following lemma was observed first by Ruckle [7, Theorem 1.20 and a remark on p. 549].

Lemma. If $X = \sum_{i=1}^\infty X_i$ is a weak decomposition then it is a Schauder decomposition.

Proof. Let $\sum_{i=1}^\infty x_j$ converge weakly to $x$, $x_j \in X_j$ for each $j$. Let $y_j = x_j/\|x_j\|$ if $x_j \neq 0$ and $y_j = 0$ if $x_j = 0$. In $[y_j]$ let $z_n = \sum_{j=1}^\infty t_{jn} y_j$ converge to $u$ as $n \to \infty$. Then, since each projection $R_j x = x_j$ is continuous $n: t_{jn} y_j \to u_j$ so that $t_{jn}$ converges, say to $t_j$, if $y_j \neq 0$ (and if $y_j = 0$ we may set each $t_{jn} = 0$). Moreover $u_j = t_j y_j$. The sequence $(t_j)$ is unique and $\sum_{i=1}^\infty t_i y_j = \sum_{i=1}^\infty u_i$ converges weakly to $u$. Thus $(y_j)$ is a weak basis for $[y_j]$ and by (K) it is a basis (this proves (i) then).

Now the representation of $x$ in $[y_j]$ is $\sum_{i=1}^\infty \|x_i\| y_j = \sum_{i=1}^\infty x_i$ which converges to $x$ as $(y_j)$ is a basis. Thus the decomposition is a Schauder decomposition (hence (ii) follows from (i) in our particular case). We remark that the argument proves such a sequence $(y_j)$ is a basis for $[y_j]$ for any space $X$ having a Schauder decomposition $\sum X_j$.

(iv) To show $[f_j]$ in (ii) is reflexive we show its unit ball is weakly compact [1, pp. 51, 56]. Let $\sum_{i=1}^\infty t_{jn} f_j = g_n \in [f_j]$ and $\|g_n\| \leq 1$ for each $n$. Use a diagonal process to find a subsequence $g_{n,i} = h_i = \sum_{j=1}^\infty s_{ji} f_j$ such
that \( s_{j_i} \to s_j \) as \( i \to \infty \) for each \( j \) (the projections \( (Q_j) \) are uniformly norm bounded so this may be done).

By (iii) \( \sum \sigma Q_j \) converges to the identity in the strong operator topology \([1, \text{p. 34}]\) so that the set \( (\sum m Q_j) \) is uniformly norm bounded, say by \( c \). Then for each \( x \) in \( B \)

\[
| h_n(x) - h_m(x) | \leq \left| \sum_{1}^{k} (s_{jn}f_j - s_{jm}f_j)(x_j) \right| + \left| \sum_{k+1}^{\infty} (s_{jn}f_j - s_{jm}f_j)(x_j) \right| \\
\leq \left| \sum_{1}^{k} (s_{jn}f_j - s_{jm}f_j)(x_j) \right| + 2c \left| \sum_{k+1}^{\infty} x_j \right| < \left| \sum_{1}^{k} (s_{jn}f_j - s_{jm}f_j)(x_j) \right| + \epsilon
\]

if \( K \) is large, \(<2\epsilon \) if \( n, m \) are large. Thus \( h_n \) is weakly* Cauchy and so weakly Cauchy. Since \( h_n(x_i) \to s_{j_n}f_i(x_i) \) as \( n \to \infty \) for every \( x_i \) in \( B_i \) one has that \( h_n(x) \to \sum s_{i}f_i(x) \) for every \( x \) in \( B \). But then \( h_n \) converges weakly to \( \sum s_{i}f_i \) in \( B^* \). Since \( \sum s_{i}f_i \) is in \([f_j]\) the unit ball of \([f_j]\) is weakly compact.

(v) Let \( F \) be in \( B^{**} \) and \( F_n = F| \sum B_j' \). Suppose \( \| F_n \| \to 0 \). Then \( \exists \epsilon > 0 \) and a subsequence \( (F_{n_i}) \) of \( (F_n) \) such that \( \| F_{n_i} \| > \epsilon \) for each \( i \).

If \( n_i > m \) then \( F_{m} = F_{n_i} \) on the domain of \( F_{n_i} \) so that \( \| F_{m} \| \geq \| F_{n_i} \| \).

Hence \( \| F_{n} \| > \epsilon \) for every \( n \). Choose \( f_1 = \sum_{1}^{n_1} g_j \) such that \( 1 \geq |f_1| \geq 1/2 \) and \( F_1(f_1) > \epsilon \). Choose \( f_2 = \sum_{n_1+1}^{n_2} g_j \) such that \( 1 \geq |f_2| \geq 1/2 \) and \( F_{n_1+1}(f_2) > \epsilon \). Continuing in this way we choose a sequence \( F_{n_j+1} \) and \( f_j = \sum_{n_j+1}^{n_{j+1}} g_j \) such that \( 1 \geq |f_j| \geq 1/2 \) and \( F_{n_j-1}(f_j) > \epsilon \) (\( n_0 = 0 \)).

Now \([g_j]\) is reflexive so by (J) \( F[|g_i| i \geq n_j] \) converges to 0 in norm. But \( F(f_j) = F_{n_j-1}(f_j) > \epsilon \) for each \( j \) contradicts \( f_j \in [g_j] j \geq n_{j-1}+1 \) for large \( j \).

(vi) To show \([b_j]\) in (i) is reflexive we prove that \([b_j]\) has a basis which is both shrinking and boundedly complete and apply (J). Choose \( f_j \) in \( B_j \) such that \( f_j(b_j) = 1 \). The identity \( I: [b_j] \to B \) has conjugate \( I^* \) such that \( I^*f_j = \phi_j \) is a biorthogonal sequence to \( (b_j) \).

Given any \( \phi \) in \([b_j]\) there is a unique sequence \( (t_j) \) such that \( \phi(x) = \sum t_j\phi_j(x) \) for each \( x \) in \([b_j]\) (let \( \phi(b_j) = t_j, x = \sum s_jb_j \), then \( \phi(x) = \sum t_j s_j = \sum t_j \phi_j(x) \)). Let \( I^*g = \phi \) and \( g = \sum g_j \). Then \( I^*g = t_j\phi_j \) for some \( t_j \) since \( I^*g_j = \sum s_jb_j \phi_j(s_jb_j) \phi_j(\sum s_jb_j) \). Thus \( t_j = g_j(b_j) \). Thus \( I^*(\sum g_j) = \sum t_j\phi_j \) converges. By the calculation preceding, it must converge to \( \phi \) so that \( (\phi_j) \) is a basis for \([b_j]\) * . If \( \| \sum t_j\phi \| \leq K \) for each \( n \) then

\[
\sum_{1}^{m} t_j\phi_j(\sum s_jb_j) \leq K
\]
or \[\sum_{j} t_j \phi_j\] converges weakly* to a limit \(\phi\) and \(\phi(b_j) = t_j\). Thus \(\phi = \sum t_j \phi_j\) so that

\[\sum_{j} t_j \phi_j \to \sum_{j} t_j \phi_j.\]

Thus \((\phi_j)\) is boundedly complete and it remains to show it is shrinking. Let \(F \in [\phi_j]^*\) and let \(F_n(\sum_j t_j \phi_j) = F(\sum_{j \geq n} t_j \phi_j)\) for every \(\sum_j t_j \phi_j\) in \([\phi_j]\). Then \(\|F_n\| \to 0\) if and only if \(\|F[\phi_j, j \geq n]\| \to 0\) as follows. The projections \(\sum_{j \geq n} t_j \phi_j \to \sum_{j \geq n} t_j \phi_j\) are uniformly norm bounded, say by \(K\). Then \(\|F[\phi_j, j \geq n]\| \leq \|F_n\| \leq K \|F[\phi_j, j \geq n]\|\). Now \(I^*F_n = I^*F\) on \(\sum_{j \geq n} B_j^\prime\) and 0 on \(\sum_{j=1}^{n-1} B_j^\prime\). Then, as above, \(\|I^*F_n\| \to 0\) if \(\|I^*F\| \sum_{j \geq n} B_j^\prime\| \to 0\) which is (v). Since \(\|I^*F_n\| = \|F_n\|\) \((I^*\) is an isometry) the basis is shrinking.

**Bibliography**


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