ON THE PROBLEM OF EVANESCENT PROCESSES

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1. Let \( R \) denote the set of real numbers. Let \( G \) be a countable dense subgroup of \( R \). We construct a nontrivial \( \sigma \)-finite measure \( m \) on \( R \) such that

(i) \( m \) is nonatomic, i.e. \( \mu(\{x\}) = 0 \) for every real number \( x \),
(ii) \( m \) is singular with respect to the Lebesgue measure \( L \) on \( R \),
(iii) \( m \) is invariant under translation by members of \( G \), i.e., \( m(A + g) = m(A) \) for all \( g \in G \) and for all Borel subsets \( A \). (Here and in sequel \( \mathcal{B} \) denotes the class of Borel subsets of \( R \).)

Such measures are intimately connected with the problem of evanescent processes and analytic functions on compact tori raised by Henry Helson and David Lowdenslager in their paper [3]. We establish this connection in §3. In §2 we shall study the group of unitary operators \( T^g \), \( g \in G \), defined on \( L^2(R, m) \) by \( (T^g f)(\lambda) = f(\lambda + g) \), \( f \in G \). In the references we list papers connected with the present work. We proceed with the construction of the measure in steps.

Step 1 (Cantor’s decimal set \( D \)). Expand every number \( x \) in the unit interval \( I = \{x = 0 \leq x < 1\} \) in the decimal system, i.e. write \( x = \sum_{n=1}^{\infty} (\alpha_n/10^n) \), \( \alpha_n = 0, 1, 2, \ldots 9 \); \( n = 1, 2, \ldots \) and let \( D \) be the set of all those numbers \( x \) in whose expansion \( \alpha_n \) takes values 0 or 9. More accurately \( D \) is the set of all numbers \( x \) in the unit interval \( I \) such that \( x \) can be expanded by using 0 and 9 alone. Geometrically \( D \) is the Cantor set obtained by deleting the middle 8/10ths.

Step 2. Here we state a known result and indicate its proof. Let \( A_5 \) denote the set of all those numbers in the unit interval \( I \) whose decimal expansion does not involve the number 5.

Lemma 1. \( A_5 \) has Lebesgue measure zero.

Proof. Let \( Q_n \) be the set of numbers \( x \in I \) such that 5 is in the \( n \)th decimal place of the expansion of \( x \) but not in the first \( (n-1) \) places. Then each \( Q_n \) is measurable and its Lebesgue measure can be shown to be \( 9^{n-1}/10^n \). \( Q_n \)’s form a disjoint sequence of sets and the Lebesgue measure of \( \bigcup_{n=1}^{\infty} Q_n \) is \( \sum_{n=1}^{\infty} (9^{n-1}/10^n) = 1 \). But \( A_5 = I - \bigcup_{n=1}^{\infty} Q_n \). So the Lebesgue measure of \( A_5 \) is zero, q.e.d

Step 3. Let \( Q = \{x-y: x, y \in D\} \).

Lemma 2. \( Q \) has Lebesgue measure zero.

Proof. \( Q = Q_+ \cup Q_- \) where

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\[ Q_+ = \{ x - y : x, y \in D, x \geq y \}, \]
\[ Q_- = \{ x - y : x, y \in D, x \leq y \} = -Q_. \]

So it is enough to show that \( Q_+ \) has Lebesgue measure zero. We shall show that \( Q_+ \subseteq A_5 \) of Lemma 1. Let \( x, y \in D \), with \( x > y \), have decimal expansions \( \cdot \alpha_1 \alpha_2 \cdots \) and \( \cdot \beta_1 \beta_2 \cdots \) respectively. Let \( x_n \) and \( y_n \) be the numbers obtained from \( x, y \) by terminating their decimal expansions at the \( n \)th stage. Then

\[ x_n - y_n = \cdot \alpha_1 \alpha_2 \cdots \alpha_n - \cdot \beta_1 \beta_2 \cdots \beta_n. \]

Since \( \alpha \)'s and \( \beta \)'s take values 0 or 9 only (since \( x, y \in D \)) it follows that \( x_n - y_n \) does not involve the number 5 in its decimal expansion. Hence \( x - y = \lim_{n \to \infty} (x_n - y_n) \) does not involve the number 5 in its decimal expansion. So the Lebesgue measure of \( Q_+ = 0 \). Hence \( L(Q) = L(Q_+) + L(Q_-) = 0 \). q.e.d.

Remark. It is interesting to note that if \( D \) were the well-known Cantor ternary set, then the set \( Q = \{ x - y : x, y \in D \} \) would be the entire interval from \(-1\) up to \(1\).

Step 4. Let \( Q \) be the set of Lebesgue measure zero of Step 3. Write
\[ F = \{ (x + m)/n : x \in Q, m, n \text{ arbitrary integers, } n \neq 0 \}. \]

Since \( Q \) has Lebesgue measure zero, \( F \) has Lebesgue measure zero. Hence there exists an irrational number \( \lambda \in F \).

Step 5. Choose an irrational number \( \lambda \in F \), where \( F \) is as in Step 4. Let \( G \) be the group \( m + \lambda n \), where \( m, n \) are integers. The group \( G \) is dense in \( R \). (A nondense subgroup of \( R \) is necessarily isomorphic to the group of integers.) Let \( D \) be the Cantor decimal set of Step 1.

Lemma 3. Translates of \( D \) by members of \( G \) are disjoint.

Proof. Let \( D + m + \lambda n, D + p + \lambda q \) be two translates of \( D \). Suppose that \( (D + m + \lambda n) \cap (D + p + \lambda q) \neq \emptyset \). Then there exists \( x, y \in D \) such that \( x + m + \lambda n = y + p + \lambda q \), i.e. \( x - y = p - m + \lambda (q - n) \). If \( q = n \), then \( p = m \) (since \( 0 \leq x, y < 1 \)) so that we do not have distinct translates. If \( q \neq n \), then \( (x - y + m - p)/(q - n) = \lambda \); but \( (x - y + m - p)/(q - n) \in F \) and \( \lambda \in F \), so we again get a contradiction. Hence translates of \( D \) by members of \( G \) are disjoint.

Step 6 (The Cantor function \( f \)). Let \( x \in I \) have the decimal expansion \( x = \cdot \alpha_1 \alpha_2 \alpha_3, \ldots, \alpha_i = 0, 1, 2, \ldots, 9 \). Let \( n = n(x) \) be the first index for which \( \alpha_n \in \{ 1, 2, \ldots, 8 \} \) and \( \alpha_n \notin \{ 0, 9 \} \). If there is no such \( n \), i.e., if \( x \in D \), write \( n(x) = \infty \). Define the function \( f \) by

\[ f(x) = \frac{1}{9} \left( \sum_{i=1}^{n-1} \frac{\alpha_i}{2^i} \right) + \frac{1}{2^n}, \quad n = n(x). \]
The function $f$ is continuous and monotonically nondecreasing with points of increase only in the set $D$ of Lebesgue measure zero.

**Step 7 (Construction of $m$).** Let $\mu$ be the finite measure induced by the monotone function $f$ of Step 6. $\mu$ is obviously nonatomic and singular with respect to the Lebesgue measure on $I$. Extend $\mu$ by setting $\mu(A) = 0$ for sets $A$ outside $I$. Let $G$ be the countable dense subgroup of Step 5 and define $m$ by

$$m(A) = \sum_{n, \lambda_n = -\infty}^{\infty} \mu(A + m + \lambda_n) = \sum_{g \in G} \mu(A + g), \quad A \in \mathcal{B}.$$ 

Clearly $m$ is invariant under translation by members of $G$. Further $m$ is nonatomic. Finally we observe that $m$ is supported on $\bigcup_{g \in G} (D + g)$, the union of countable number of disjoint sets $D + g$, $g \in G$, and that $m(D + g) = m(D) = 1$. Hence $m$ is $\sigma$-finite. This completes the construction of $m$.

**Remark.** We have constructed the measure $m$ invariant under translation by a dense subgroup with two generators. But this is not a restriction. With little manipulation one can construct a $\sigma$-finite nonatomic singular measure invariant under translation by any countable subgroup of the real line.

2. From now on we shall denote by $G$ a fixed countable dense subgroup of $\mathbb{R}$. A measure $m$ on $\mathcal{B}$ is called nonatomic singular $G$-invariant if

(i) $m$ is nonatomic,
(ii) $m$ is singular with respect to the Lebesgue measure on $\mathbb{R}$,
(iii) $m(A + g) = m(A)$ for all $A \in \mathcal{B}$ and $g \in G$,
(iv) There exists a Borel set $D$ of finite $m$ measure such that the translates $D + g$, $g \in G$, of $D$ by members of $G$ are pair wise disjoint and $\bigcup_{g \in G} D + g$ supports $m$.

A method of constructing such measures was given in §1.

Now fix a continuous singular $G$-invariant measure $m$ on $\mathbb{R}$. Let $L_2(\mathbb{R}, m)$ be the linear space of functions square integrable with respect to $m$. Let $D$ be the set (the existence of which is guaranteed by (iv)) such that the sets $D + g = D_g$ are pair wise disjoint and $\bigcup_{g \in G} D_g$ supports $m$. Then clearly $L_2(\mathbb{R}, m) = \sum_{g \in G} \mathbb{D} L_2(D_g, m)$ where $L_2(D_g, m)$ is the set of functions in $L_2(\mathbb{R}, m)$ that vanish outside $D_g$. The orthogonal projection of $f \in L_2(\mathbb{R}, m)$ on $L_2(D_g, m)$ is given by $f I_g$, where $I_g$ is the characteristic function of $D_g$.

Now $m$ is $G$-invariant so we get a group $T^g$, $g \in G$, of unitary operators defined on $L_2(\mathbb{R}, m)$ by $(T^g f)(\lambda) = f(\lambda + g)$, $f \in L_2(\mathbb{R}, m)$, $g \in G$.

Let us define a spectral measure $E$ on $\mathcal{B}$ by writing $E(\sigma)f = I_{\sigma} f$, $f \in L_2(\mathbb{R}, m)$ where $I_{\sigma}$ is the characteristic function of $\sigma$. It is easily
varified that $E$ and $T^g$ are connected by the relation $T^gE(\sigma)T^{-g} = E(\sigma - g)$ for all $g \in G$. But $T^g$ is not the only commutative group of unitary operators which satisfies this equation with $E$. The general commutative group $U^g$, which with $E$, satisfies $U^gE(\sigma)U^{-g} = E(\sigma - g)$, $g \in G$, has the following form. $U^g$ is defined by

$$(U^g f)(\lambda) = A(g, \lambda)f(\lambda + g), \quad g \in G, f \in L^2(R, m),$$

where $A(g, \lambda)$ is an $m$-measurable function of $\lambda$ for every fixed $g$ such that

(i) $|A(g, \lambda)| = 1$,

(ii) $A(g + h, \lambda) = A(g, \lambda)A(h, \lambda + g)$ for almost all $\lambda$ with respect to the $m$-measure.

The set of $m$-measure zero where (ii) does not hold may vary with the pair $(g, h)$.

Functions satisfying the functional equation (ii) occur very crucially in the study of spectral measures $E$ on $\mathfrak{B}$ for which there exists a commutative group $U^g$, $g \in G$, satisfying the equation $U^gE(\sigma)U^{-g} = E(\sigma + g)$, $g \in G$, $\sigma \in \mathfrak{B}$ (cf. §4).

The group $U^g$ has a spectral measure associated with it as follows. (See [6, p. 392].)

Let $B = \hat{G}_d$ be the compact dual of $G_d$, the group $G$ with the discrete topology. Since $U^g$ is a commutative group of unitary operators, by Godement’s extension of Stone’s theorem on the representation of unitary operators [1] there exists a Hermitian projection valued spectral measure $F$ on the Borel subsets $\mathfrak{F}$ of $B$ such that $U^g = \int_B \chi_g(\lambda)dF_\lambda$ in the sense that

$$(U^g f, h) = \int_B \chi_g(\lambda)(dF_\lambda f, h), \quad f, h \in L^2(R, m).$$

Here $\chi_g$ denotes the character on $B$ corresponding to $g \in G_d$. For $f, h \in L^2(R, m)$, $(F(\cdot)f, h)$ defines a complex valued finite measure on $\mathfrak{F}$ so that for $\sigma \in \mathfrak{F}$ the value of this measure is $(F(\sigma)f, h)$.

We show that for every $f$ and $h$, $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on $B$.

**Theorem 1.** If $f \in L^2(D_{g_0}, m)$ for some $g_0 \in G$, then the measure $(F(\cdot)f, f)$ is a constant multiple of the Haar measure on $B$. For any $f, h \in L^2(R, m)$, the measure $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on $B$.

**Proof.** Let $f \in L^2(D_{g_0}, m)$, then $U^g f \in L^2(D_{g_0} - g, m)$. Hence, the elements $\{U^g f : g \in G\}$ are mutually orthogonal. Now by (*) $(U^g f, f) = \int_B \chi_g(\lambda)(dF_\lambda f, f) = 0$ if $g \neq 0$. Hence $(F(\cdot)f, f)$ is a constant multiple
of the Haar measure on $B$. The constant multiple is, of course, non-zero if and only if $f \neq 0$ in $L_2(D_{g_0}, m)$. Now let $f \in L_2(D_{g_0}, m)$, $h \in L_2(D_{g_0}, m)$, then by the polarization formula it is easy to see that $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on $B$. Finally choose any $f, h \in L_2(R, m)$. Let $f = \sum_{g \in G} f_g, h = \sum_{g \in G} h_g, f_g, h_g \in L_2(D_g, m)$. Then clearly

$$ (F(\cdot)f, h) = \sum_{g, g' \in G} (F(\cdot)f_g, h_{g'}). $$

Since each $(F(\cdot)f_g, h_{g'})$ is absolutely continuous with respect to the Haar measure on $B$, it follows that $(F(\cdot)f, h)$ has the same property, q.e.d.

The next theorem shows that Wiener closure theorem has no analogue for a nonatomic singular $G$-invariant measure.

**Theorem 2.** There is no $f \in L_2(R, m)$ such that \{ $U^g f \cdot g \in G$ \} spans $L_2(R, m)$.

To prove this theorem we need a known result which we state here without proof for the sake of completeness.

**Lemma 4.** Let $\mu$ be a finite positive regular measure on the Borel subsets $\mathcal{F}$ of $B$. Let $h, f \in L_2(B, \mu)$. Then $\int_B \chi_g(\lambda) h(\lambda) f(\lambda) \, d\mu = 0$ for all $g \in G_d$ if and only if $h$ vanishes almost everywhere with respect to $\mu$ on the set where $|f| > 0$.

This lemma is an easy consequence of the fact that a finite regular Borel measure on a locally compact abelian group is uniquely determined by its Fourier-Stieltjes transform [7, p. 17].

Consider the measures on $\mathfrak{F}$ defined by $\int_B \chi_g(\lambda) h(\lambda) \, d\mu$, $\int_B |f(\lambda)|^2 \, d\mu$. Then the lemma is equivalent to the following fact:

$$ \int_B \chi_g(\lambda) h(\lambda) \overline{f(\lambda)} \, d\mu = 0 $$

for all $g \in G_d$ if and only if the measures $\int_B \chi_g(\lambda) h(\lambda) \, d\mu$ and $\int_B |f(\lambda)|^2 \, d\mu$ are mutually singular.

**Proof of Theorem 2.** Suppose that there exists $f \in L_2(R, m)$ such that \{ $U^g f \cdot g \in G$ \} spans $L_2(R, m)$. By $(\star)$, $(U^g f, f) = \int_B \chi_g(\lambda) (dF_{\lambda}f, f) = \int_B \chi_g(\lambda) \, d\mu$, where $\mu$ is the measure defined by $\mu(\sigma) = (F(\cdot)f, f)$, $\sigma \in \mathfrak{F}$. By Theorem 1, $\mu$ is absolutely continuous with respect to the Haar measure on $B$. The mapping $S$: $SU^g = \chi_g$ extends by linearity to an invertible isometry from the space spanned by \{ $U^g f$: $g \in G$ \} to $L_2(B, \mu)$. Now let $D_1, D_2$ be two disjoint measurable subsets of $D$.
such that $D = D_1 \cup D_2$ and $m(D_1), m(D_2) > 0$. Let $h_1$ and $h_2$ denote the characteristic functions of $D_1$ and $D_2$. Write $f_1 = Sh_1$, $f_2 = Sh_2$. It is clear that

(i) $U^g h_1$ are all mutually orthogonal in $L_2(R, m)$,
(ii) $U^g h_2$ are all mutually orthogonal in $L_2(R, m)$,
(iii) $U^g h_1 \perp U^{g'} h_2$ for all $g, g' \in G$.

This is because the translates of $D$ by members of $G$ are disjoint.

Now for all $g \in G$, $(U^g h_1, h_1) = \int_{\chi_g} d\mu = 0$ if $g \neq 0$. Similarly $(U^g h_2, h_2) = \int_{\chi_g} d\mu = 0$ for all $g$. Hence by (iii) $(U^g h_1, h_2) = 0$ for all $g, g'$ by (iii). Hence $(U^g h_1, h_2) = 0$ for all $g$. Hence by Lemma 4 the measures $\mathcal{F}(\cdot, h_1, h_1)$ and $\mathcal{F}(\cdot, h_2, h_2)$ are mutually singular. This is a contradiction, q.e.d.

3. In this section we show how the measures of the type discussed in §2 are excluded in the problem of evanescent processes. First we must explain this problem.

Let $G$ and $B$ be as in §2. Let $f$ be a nonzero positive function on $B$ summable with respect to the Haar measure on $B$. Let $L_2(B, f) = \{ \psi : |\psi|^2 f \text{ is summable with respect to the Haar measure on } B \}$. Let $H_t$ be the subspace of $L_2(B, f)$ spanned by $\{ \chi_g : g < t \}$, where $\chi_g$ denotes the character on $B$ corresponding to the real number $g \in G$. It is clear that $H_t \subseteq H_{t'}$ whenever $t < t'$. It can be shown that either $H_t = H_{t'}$ for all $t, t'$ or $\bigcap_t H_t = \{ 0 \}$ and $H_t \nsubseteq H_{t'}$ whenever $t < t'$. This has been shown by Helson and Lowdenslager in their paper [3]. The problem of evanescent processes can be stated as follows: Assume that $H_t \neq H_{t'}$ for $t < t'$, then is it always true that $(\bigcap_{t > 0} H_t) \ominus H_0 \neq \{ 0 \}$?

A well-known result of Helson and Lowdenslager [2] answers the question in the affirmative under the assumption that $\log f$ is summable with respect to the Haar measure on $B$. In what follows we give further evidence in favor of the affirmative answer to the question.

The increasing subspaces $H_t$ give rise to a spectral measure $E$ on the Borel subsets of $R$. For intervals $(a, b]$, $E$ is given by $E(a, b] = \text{orthogonal projection on } H_b \ominus H_a$. In $L_2(B, f)$ there is a commutative group $U^g$ of unitary operators defined by $U^g \psi = \chi_g \psi$, $\psi \in L_2(B, f)$, $g \in G$. Further the following two identities are easily verified

(A) $U^g (H_b \ominus H_a) = H_{b+g} \ominus H_{a+g}$ where $a, b$ ($a < b$) are any two real numbers.

(B) For any $\psi \in L_2(B, f)$, $\| E(a, b] \psi - \psi \|^2 = \| U^g E(a, b] \psi - U^g \psi \|^2$. 

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(A) and (B) together imply that $U^o$ and $E$ are connected by the relation $U^oE(\sigma)U^{-o}=E(\sigma+g)$ for all $\sigma \in G$ and $g \in G$. Helson and Lowdenslager have shown that if $E\{x\} \neq 0$ for some $x$, then the spectral measure $E$ is purely discrete and $E$ has no continuous component. Now it can be shown that $E$ cannot have a component absolutely continuous with respect to the Lebesgue measure on $R$, i.e., there does not exist a nonzero $\psi \in L_c(B, f)$ such that $(E(\cdot)\psi, \psi)$ is absolutely continuous with respect to the Lebesgue measure on $R$. In what follows we show that $E$ has no component absolutely continuous with respect to a nonatomic singular $G$-invariant measure on $R$.

**Theorem 3.** Assume that $E\{x\} = 0$ for all $x$. There does not exist a Borel set $D$ such that:

(i) the sets $D+g$, $g \in G$ are mutually disjoint,

(ii) $E(D) \neq 0$.

**Proof.** Suppose not. Then there exists a set $D$ such that the sets $D+g$, $g \in G$, are mutually disjoint and $E(D) \neq 0$. Since $E$ has no discrete spectrum, we can find two nonzero vectors $\Phi, \psi$ in $E(D)$ such that $\Phi$ and $\psi$ are mutually orthogonal. Now $U^o\Phi = U^oE(D)\Phi = E(D+g)U^o\Phi \in E(D+g)$ and similarly $U^o\psi \in E(D+g)$. Since the sets $D+g$, $g \in G$, are mutually disjoint, we see that $U^o\Phi \perp \Phi$, $U^o\psi \perp \psi$ for all $g \neq 0$ and $U^o\Phi \perp U^o\psi$ for all $g, g'$. So

(i) $(U^o\Phi, \Phi) = \int_B \chi_\sigma(\lambda) |\Phi(\lambda)|^2 f(\lambda) d\sigma = 0$ for $g \neq 0$.

(ii) $(U^o\psi, \psi) = \int_B \chi_\sigma(\lambda) |\psi(\lambda)|^2 f(\lambda) d\sigma = 0$ for $g \neq 0$.

(iii) $(U^o\Phi, \psi) = \int_B \chi_\sigma(\lambda) \Phi(\lambda) \psi(\lambda) f(\lambda) d\sigma = 0$ for all $g$.

(Here $d\sigma$ is the normalized Haar measure on $B$.)

The first two equations above say that $|\Phi|^2 f d\sigma$ and $|\psi|^2 f d\sigma$ are nonzero constant multiples of the Haar measure on $B$ and the third equation says that $\Phi \psi f$ is equal to zero almost everywhere with respect to the Haar measure on $B$. This is impossible, q.e.d.

4. Let $E$ be a spectral measure on the Borel subsets of $R$ and let $G$ be a countable dense subgroup of $R$. We call a spectral measure $E$ $G$-stationary if there exists a commutative group $U^o$ of unitary operators such that $U^oE(\sigma)U^{-o}=E(\sigma+g)$ for all $\sigma \in G$ and $g \in G$. If one tries to obtain the canonical representation of $G$-stationary spectral measures like the one there is for a pair of commutative groups of unitary operators satisfying Weyl's commutativity relation one at once faces the following question.

Let $\mu$ be a finite positive measure on $G$. Call $\mu$ $G$-quasi invariant if $\mu$ and $\mu_g$ are mutually absolutely continuous for all $g \in G$. Here $\mu_g$ is defined by $\mu_g(A) = \mu(A+g)$, $A \in G$, $g \in G$.
Question 1. \( \mu \) is \( G \)-quasi invariant. Does there exist a \( \sigma \)-definite measure \( m \) on \( \mathcal{B} \) such that (i) \( m(\sigma + g) = m(\sigma) \) for all \( \sigma \in \mathcal{B}, \ g \in G \), (ii) \( m \) and \( \mu \) are mutually absolutely continuous?

Now suppose that \( \mu = \mu^d + \mu^a + \mu^s \) where \( \mu^d \) is the atomic part of \( \mu \), \( \mu^a \) is part of \( \mu \) absolutely continuous with respect to the \( L \), the Lebesgue measure, and \( \mu^s \) is nonatomic singular part of \( \mu \). It is easy to see that each component \( \mu^d \), \( \mu^a \) and \( \mu^s \) is separately \( G \)-quasi invariant.

Further \( \mu^a \) and the Lebesgue measure are mutually absolutely continuous. Thus for \( \mu^a \) the question raised above has a solution. One can also show easily that the question raised above has a solution for \( \mu^d \). Hence in the question raised above one can assume that \( \mu \) is nonatomic singular measure.

We give a reformulation of our question in terms of the functions \( A(g, \lambda) = \frac{d\mu_g}{d\mu}(\lambda) \). One verifies very easily that \( A(g, \lambda) \) satisfy the relation \( A(g+h, \lambda) = A(g, \lambda)A(h, \lambda+g) \) a.e. \([\mu]\).

**Theorem 4.** Question 1 has a solution if and only if there exists a measurable function \( B \) such that \( A(g, \lambda) = B(\lambda+g)/B(\lambda) \).

**Proof.** Suppose there exists an \( m \) as in Question 1. Write \( B(\lambda) = \frac{d\mu_0}{d\mu}(\lambda) \). Then clearly \( A(g, \lambda) = \frac{d\mu_0}{d\mu}(\lambda) = \frac{d\mu_0}{d\mu}(\lambda) \cdot \frac{d\mu}{d\mu}(\lambda) = \frac{d\mu_0}{d\mu}(\lambda) \cdot 1/B(\lambda) \). Now by the invariance of \( m \) under translation by \( g \) it is easy to see that \( \frac{d\mu_0}{d\mu}(\lambda) = B(\lambda+g) \); thus \( A(g, \lambda) = B(\lambda+g)/B(\lambda) \). Conversely suppose that \( A(g, \lambda) = B(\lambda+g)/B(\lambda) \) where \( B \) is measurable. Define \( m \) by \( m(\sigma) = \int_\sigma [B(\lambda)]^{-1}d\mu \). It is clear that \( m \) and \( \mu \) are mutually absolutely continuous. Next to see the \( G \)-invariance of \( m \) we note that

\[
m(\sigma + g) = \int_{\sigma + g} [B(\lambda)]^{-1}d\mu = \int_\sigma [B(\lambda + g)]^{-1}d\mu_0(\lambda)
\]

\[
= \int_\sigma [B(\lambda + g)]^{-1} \frac{d\mu_0}{d\mu}(\lambda)d\mu = \int_\sigma \left[\frac{B(\lambda + g)}{B(\lambda)}\right]^{-1} B(\lambda + g) d\mu
\]

\[
= \int [B(\lambda)]^{-1}d\mu = m(\sigma), \quad \text{q.e.d.}
\]

We conclude by making the following remarks.

Assume that \( \mu \) of Question 1 is singular. In order that Question 1 have an affirmative solution it is enough that there is a \( \mu \)-measurable set \( D \) such that \( D + g, \ g \in G \) are disjoint and \( \bigcup_{g \in G} (D + g) \) supports \( \mu \). However, there exist singular \( G \)-quasi invariant measures for which no such \( D \) exists. We illustrate this by the following example. Let \( C \)
be the Cantor ternary set and \( \psi \) the Cantor function from \( C \) onto \([0, 1]\). \( \psi \) is strictly increasing and continuous on \( C \) with range \([0, 1]\).

Let \( P \) be the singular measure on the real line induced by \( \psi \). Let \( G \) be the group of real members having finitely many terms in their ternary expansions. Let \( g_1, g_2, g_3, \ldots \) be a denumeration of \( G \). Write

\[
\mu(A) = \sum_{n=1}^{\infty} (1/2^n) P(A + g_n), \quad A \in \mathcal{B}.
\]

Clearly \( \mu \) is \( G \)-quasi invariant.

Call two members of \( C \) equivalent if their difference belongs to \( G \). This equivalence relation partitions \( C \). Choose a member from each equivalence class and call the new set \( D \). Translates \( D + g, g \in G \) are disjoint and \( \bigcup_{g \in G} (D + g) \) supports \( \mu \). But \( D \) can never be chosen to be \( \mu \)-measurable, for the difference of two members of \( \psi(D) \) has always finite binary expansions, so that \( \psi(D) \) is nonmeasurable. Hence \( D = \psi^{-1}(\psi(D)) \) is non-\( \mu \)-measurable.

References