# ON THE PROBLEM OF EVANESCENT PROCESSES

Z. DITZIAN AND M. NADKARNI

1. Let $R$ denote the set of real numbers. Let $G$ be a countable dense subgroup of $R$. We construct a nontrivial $\sigma$-finite measure $m$ on $R$ such that

(i) $m$ is nonatomic, i.e. $\mu(\{x\})=0$ for every real number $x$,

(ii) $m$ is singular with respect to the Lebesgue measure $L$ on $R$,

(iii) $m$ is invariant under translation by members of $G$, i.e., $m(A+g)=m(A)$ for all $g\in G$ and for all Borel subsets $A\in\mathcal{B}$. (Here and in sequel $\mathcal{B}$ denotes the class of Borel subsets of $R$.)

Such measures are intimately connected with the problem of evanescent processes and analytic functions on compact tori raised by Henry Helson and David Lowdenslager in their paper [**3**]. We establish this connection in §3. In §2 we shall study the group of unitary operators $T^g$, $g\in G$, defined on $L_2(R, m)$ by $(T^g f)(\lambda)=f(\lambda+g)$, $f\in G$. In the references we list papers connected with the present work. We proceed with the construction of the measure in steps.

*Step* 1 (*Cantor's decimal set* $D$). Expand every number $x$ in the unit interval $I=\{x=0\leqq x<1\}$ in the decimal system, i.e. write $x=\sum_{n=1}^{\infty}(\alpha_n/10^n)$, $\alpha_n=0, 1, 2, \cdots 9$; $n=1, 2, \cdots$ and let $D$ be the set of all those numbers $x$ in whose expansion $\alpha_n$ takes values 0 or 9. More accurately $D$ is the set of all numbers $x$ in the unit interval $I$ such that $x$ can be expanded by using 0 and 9 alone. Geometrically $D$ is the Cantor set obtained by deleting the middle 8/10ths.

*Step* 2. Here we state a known result and indicate its proof. Let $A_5$ denote the set of all those numbers in the unit interval $I$ whose decimal expansion does not involve the number 5.

LEMMA 1. *$A_5$ has Lebesgue measure zero.*

PROOF. Let $Q_n$ be the set of numbers $x\in I$ such that 5 is in the $n$th decimal place of the expansion of $x$ but not in the first $(n-1)$ places. Then each $Q_n$ is measurable and its Lebesgue measure can be shown to be $9^{n-1}/10^n$. $Q_n$'s form a disjoint sequence of sets and the Lebesgue measure of $\bigcup_{n=1}^{\infty} Q_n$ is $\sum_{n=1}^{\infty}(9^{n-1}/10^n)=1$. But $A_5=I-\bigcup_{n=1}^{\infty} Q_n$. So the Lebesgue measure of $A_5$ is zero, q.e.d

*Step* 3. Let $Q=\{x-y: x, y\in D\}$.

LEMMA 2. *$Q$ has Lebesgue measure zero.*

PROOF. $Q=Q_+\cup Q_-$ where

Received by the editors July 18, 1966.

$$Q_+ = \{x - y: x, y \in D, x \geqq y\},$$
$$Q_- = \{x - y: x, y \in D, x \leqq y\} = -Q_+.$$

So it is enough to show that $Q_+$ has Lebesgue measure zero. We shall show that $Q_+ \subset A_5$ of Lemma 1. Let $x, y \in D$, with $x > y$, have decimal expansions $\cdot\alpha_1\alpha_2 \cdots$ and $\cdot\beta_1\beta_2 \cdots$ respectively. Let $x_n$ and $y_n$ be the numbers obtained from $x$, $y$ by terminating their decimal expansions at the $n$th stage. Then

$$x_n - y_n = \cdot\alpha_1\alpha_2 \cdots \alpha_n - \cdot\beta_1\beta_2 \cdots \beta_n.$$

Since $\alpha$'s and $\beta$'s take values 0 or 9 only (since $x, y \in D$) it follows that $x_n - y_n$ does not involve the number 5 in its decimal expansion. Hence $x - y = \lim_{n\to\infty}(x_n - y_n)$ does not involve the number 5 in its decimal expansion. So the Lebesgue measure of $Q_+ = 0$. Hence $L(Q) = L(Q_+) + L(Q_-) = 0$. q.e.d.

REMARK. It is interesting to note that if $D$ were the well-known Cantor ternary set, then the set $Q = \{x - y \cdot x, y \in D\}$ would be the entire interval from $-1$ up to 1.

*Step* 4. Let $Q$ be the set of Lebesgue measure zero of Step 3. Write $F = \{(x+m)/n: x \in Q, m, n \text{ arbitrary integers}, n \neq 0\}$.

Since $Q$ has Lebesgue measure zero, $F$ has Lebesgue measure zero. Hence there exists an irrational number $\lambda \notin F$.

*Step* 5. Choose an irrational number $\lambda \notin F$, where $F$ is as in Step 4. Let $G$ be the group $m + \lambda n$, where $m$, $n$ are integers. The group $G$ is dense in $R$. (A nondense subgroup of $R$ is necessarily isomorphic to the group of integers.) Let $D$ be the Cantor decimal set of Step 1.

LEMMA 3. *Translates of $D$ by members of $G$ are disjoint.*

PROOF. Let $D + m + \lambda n$, $D + p + \lambda q$ be two translates of $D$. Suppose that $(D + m + \lambda n) \cap (D + p + \lambda q) \neq \emptyset$. Then there exists $x, y \in D$ such that $x + m + \lambda n = y + p + \lambda q$, i.e. $x - y = p - m + \lambda(q - n)$. If $q = n$, then $p = m$ (since $0 \leqq x, y < 1$) so that we do not have distinct translates. If $q \neq n$, then $(x - y + m - p)/(q - n) = \lambda$; but $(x - y + m - p)/(q - n) \in F$ and $\lambda \notin F$, so we again get a contradiction. Hence translates of $D$ by members of $G$ are disjoint.

*Step* 6 (*The Cantor function $f$*). Let $x \in I$ have the decimal expansion $x = \cdot\alpha_1\alpha_2\alpha_3, \cdots, \alpha_i = 0, 1, 2, \cdots, 9$. Let $n = n(x)$ be the first index for which $\alpha_n \in \{1, 2, \cdots, 8\}$ and $\alpha_n \notin \{0, 9\}$. If there is no such $n$, i.e., if $x \in D$, write $n(x) = \infty$. Define the function $f$ by

$$f(x) = \frac{1}{9}\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{2^i}\right) + \frac{1}{2^n}, \qquad n = n(x).$$

The function $f$ is continuous and monotonically nondecreasing with points of increase only in the set $D$ of Lebesgue measure zero.

*Step* 7 (*Construction of* $m$). Let $\mu$ be the finite measure induced by the monotone function $f$ of Step 6. $\mu$ is obviously nonatomic and singular with respect to the Lebesgue measure on $I$. Extend $\mu$ by setting $\mu(A)=0$ for sets $A$ outside $I$. Let $G$ be the countable dense subgroup of Step 5 and define $m$ by

$$m(A) = \sum_{m,n=-\infty}^{\infty} \mu(A + m + \lambda_n) = \sum_{g\in G} \mu(A + g), \qquad A \in \mathcal{B}.$$

Clearly $m$ is invariant under translation by members of $G$. Further $m$ is nonatomic. Finally we observe that $m$ is supported on $\bigcup_{g\in G}(D+g)$, the union of countable number of disjoint sets $D+g$, $g\in G$, and that $m(D+g)=m(D)=1$. Hence $m$ is $\sigma$-finite. This completes the construction of $m$.

REMARK. We have constructed the measure $m$ invariant under translation by a dense subgroup with two generators. But this is not a restriction. With little manipulation one can construct a $\sigma$-finite nonatomic singular measure invariant under translation by any countable subgroup of the real line.

2. From now on we shall denote by $G$ a fixed countable dense subgroup of $R$. A measure $m$ on $\mathcal{B}$ is called nonatomic singular $G$-invariant if

(i) $m$ is nonatomic,

(ii) $m$ is singular with respect to the Lebesgue measure on $R$,

(iii) $m(A+g)=m(A)$ for all $A\in\mathcal{B}$ and $g\in G$,

(iv) There exists a Borel set $D$ of finite $m$ measure such that the translates $D+g$, $g\in G$, of $D$ by members of $G$ are pair wise disjoint and $\bigcup_{g\in G}(D+g)$ supports $m$.

A method of constructing such measures was given in §1.

Now fix a continuous singular $G$-invariant measure $m$ on $R$. Let $L_2(R, m)$ be the linear space of functions square integrable with respect to $m$. Let $D$ be the set (the existence of which is guaranteed by (iv)) such that the sets $D+g=D_g$ are pair wise disjoint and $\bigcup_{g\in G} D_g$ supports $m$. Then clearly $L_2(R, m)=\sum_{g\in G}\oplus L_2(D_g, m)$ where $L_2(D_g, m)$ is the set of functions in $L_2(R, m)$ that vanish outside $D_g$. The orthogonal projection of $f\in L_2(R, m)$ on $L_2(D_g, m)$ is given by $fI_g$, where $I_g$ is the characteristic function of $D_g$.

Now $m$ is $G$-invariant so we get a group $T^g$, $g\in G$, of unitary operators defined on $L_2(R, m)$ by $(T^g f)(\lambda)=f(\lambda+g)$, $f\in L_2(R, m)$, $g\in G$.

Let us define a spectral measure $E$ on $\mathcal{B}$ by writing $E(\sigma)f=I_\sigma f$, $f\in L_2(R, m)$ where $I_\sigma$ is the characteristic function of $\sigma$. It is easily

varified that $E$ and $T^g$ are connected by the relation $T^g E(\sigma) T^{-g} = E(\sigma - g)$ for all $g \in G$. But $T^g$ is not the only commutative group of unitary operators which satisfies this equation with $E$. The general commutative group $U^g$, which with $E$, satisfies $U^g E(\sigma) U^{-g} = E(\sigma - g)$, $g \in G$, has the following form. $U^g$ is defined by

$$(U^g f)(\lambda) = A(g, \lambda) f(\lambda + g), \qquad g \in G, f \in L_2(R, m),$$

where $A(g, \lambda)$ is an $m$-measurable function of $\lambda$ for every fixed $g$ such that

(i) $|A(g, \lambda)| = 1$,

(ii) $A(g+h, \lambda) = A(g, \lambda) A(h, \lambda + g)$ for almost all $\lambda$ with respect to the $m$-measure.

The set of $m$-measure zero where (ii) does not hold may vary with the pair $(g, h)$.

Functions satisfying the functional equation (ii) occur very crucially in the study of spectral measures $E$ on $\mathcal{B}$ for which there exists a commutative group $U^g$, $g \in G$, satisfying the equation $U^g E(\sigma) U^{-g} = E(\sigma + g)$, $g \in G$, $\sigma \in \mathcal{B}$ (cf. §4).

The group $U^g$ has a spectral measure associated with it as follows. (See [6, p. 392].)

Let $B = \hat{G}_d$ be the compact dual of $G_d$, the group $G$ with the discrete topology. Since $U^g$ is a commutative group of unitary operators, by Godement's extension of Stone's theorem on the representation of unitary operators [1] there exists a Hermitian projection valued spectral measure $F$ on the Borel subsets $\mathfrak{F}$ of $B$ such that $U^g = \int_B \chi_g(\lambda) dF_\lambda$ in the sense that

$$(*) \qquad (U^g f, h) = \int_B \chi_g(\lambda)(dF_\lambda f, h), \qquad f, h \in L_2(R, m).$$

Here $\chi_g$ denotes the character on $B$ corresponding to $g \in G_d$. For $f, h \in L_2(R, m)$, $(F(\cdot)f, h)$ defines a complex valued finite measure on $\mathfrak{F}$ so that for $\sigma \in \mathfrak{F}$ the value of this measure is $(F(\sigma)f, h)$.

We show that for every $f$ and $h$, $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on $B$.

THEOREM 1. *If $f \in L_2(D_{g_0}, m)$ for some $g_0 \in G$, then the measure $(F(\cdot)f, f)$ is a constant multiple of the Haar measure on $B$. For any $f, h \in L_2(R, m)$, the measure $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on $B$.*

PROOF. Let $f \in L_2(D_{g_0}, m)$, then $U^g f \in L_2(D_{g_0} - g, m)$. Hence, the elements $\{U^g f : g \in G\}$ are mutually orthogonal. Now by $(*)$ $(U^g f, f) = \int_B \chi_g(\lambda)(dF_\lambda f, f) = 0$ if $g \neq 0$. Hence $(F(\cdot)f, f)$ is a constant multiple

of the Haar measure on $B$. The constant multiple is, of course, nonzero if and only if $f\neq 0$ in $L_2(D_{g_0}, m)$. Now let $f\in L_2(D_{g_0}, m)$, $h\in L_2(D_{g_1}, m)$, then by the polarization formula it is easy to see that $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on $B$. Finally choose any $f,h\in L_2(R, m)$. Let $f=\sum_{g\in G} f_g$, $h=\sum_{g\in G} h_g$, $f_g$, $h_g\in L_2(D_g, m)$. Then clearly

$$(F(\cdot)f, h) = \sum_{g,g'\in G} (F(\cdot)f_g, h_{g'}).$$

Since each $(F(\cdot)f_g, h_{g'})$ is absolutely continuous with respect to the Haar measure on $B$, it follows that $(F(\cdot)f, h)$ has the same property, q.e.d.

The next theorem shows that Wiener closure theorem has no analogue for a nonatomic singular $G$-invariant measure.

THEOREM 2. *There is no $f\in L_2(R, m)$ such that $\{U^g f \cdot g\in G\}$ spans $L_2(R, m)$.*

To prove this theorem we need a known result which we state here without proof for the sake of completeness.

LEMMA 4. *Let $\mu$ be a finite positive regular measure on the Borel subsets $\mathfrak{F}$ of $B$. Let $h, f\in L_2(B, \mu)$. Then $\int_B \chi_g(\lambda)h(\lambda)\bar{f}(\lambda)d\mu=0$ for all $g\in G_d$ if and only if $h$ vanishes almost everywhere with respect to $\mu$ on the set where $|f|>0$.*

This lemma is an easy consequence of the fact that a finite regular Borel measure on a locally compact abelian group is uniquely determined by its Fourier-Stieltjes transform [7, p. 17].

Consider the measures on $\mathfrak{F}$ defined by $\int_\sigma |h(\lambda)|^2 d\mu$, $\int_\sigma |f(\lambda)|^2 d\mu$. Then the lemma is equivalent to the following fact:

$$\int_B \chi_g(\lambda)h(\lambda)\overline{f(\lambda)}d\mu = 0$$

for all $g\in G_d$ if and only if the measures $\int_{(\cdot)} |h(\lambda)|^2 d\mu$ and $\int_{(\cdot)} |f(\lambda)|^2 d\mu$ are mutually singular.

PROOF OF THEOREM 2. Suppose that there exists $f\in L_2(R, m)$ such that $\{U^g f\cdot g\in G\}$ spans $L_2(R, m)$. By (*), $(U^g f, f)=\int \chi_g(\lambda)(dF_\lambda f, f) =\int \chi_g(\lambda)d\mu$, where $\mu$ is the measure defined by $\mu(\sigma)=(F(\sigma)f, f)$, $\sigma\in\mathfrak{F}$. By Theorem 1, $\mu$ is absolutely continuous with respect to the Haar measure on $B$. The mapping $S$: $SU^g f=\chi_g$ extends by linearity to an invertible isometry from the space spanned by $\{U^g f: g\in G\}$ to $L_2(B, \mu)$. Now let $D_1$, $D_2$ be two disjoint measureable subsets of $D$

such that $D=D_1\cup D_2$ and $m(D_1)$, $m(D_2)>0$. Let $h_1$ and $h_2$ denote the characteristic functions of $D_1$ and $D_2$. Write $f_1=Sh_1$, $f_2=Sh_2$. It is clear that

(i) $U^g h_1$ are all mutually orthogonal in $L_2(R, m)$,

(ii) $U^g h_2$ are all mutually orthogonal in $L_2(R, m)$,

(iii) $U^g h_1 \perp U^{g'} h_2$ for all $g$, $g'\in G$.

This is because the translates of $D$ by members of $G$ are disjoint.

Now for all $g\in G$, $(U^g h_1, h_1)=\int \chi_g(\lambda)|f_1(\lambda)|^2 d\mu=\int_B \chi_g(\lambda)(dF_\lambda h_1, h_1)=0$ if $g\neq 0$. Similarly $(U^g h_2, h_2)=\int \chi_g(\lambda)|f_2(\lambda)|^2 d\mu=\int_B \chi_g(\lambda)(dF_\lambda h_2, h_2)=0$.

Hence the measures $(F(\cdot)h_1, h_1)$ and $(F(\cdot)h_2, h_2)$ are nonzero constant multiples of the Haar measure on $B$. Hence the measures $\int_{(\cdot)}|f_1(\lambda)|^2 d\mu$ and $\int_{(\cdot)}|f_2(\lambda)|^2 d\mu$ are nonzero constant multiples of the Haar measure on $B$. But $U^g h_1 \perp U^{g'} h_2$ for all $g$, $g'$ by (iii). Hence $(U^g h_1, h_2)=\int_B \chi_g f_1 \bar{f}_2 d\mu=0$ for all $g$. Hence by Lemma 4 the measures $\int_{(\cdot)}|f_1(\lambda)|^2 d\mu$ and $\int_{(\cdot)}|f_2(\lambda)|^2 d\mu$ are mutually singular. This is a contradiction, q.e.d.

3. In this section we show how the measures of the type discussed in §2 are excluded in the problem of evanescent processes. First we must explain this problem.

Let $G$ and $B$ be as in §2. Let $f$ be a nonzero positive function on $B$ summable with respect to the Haar measure on $B$. Let $L_2(B, f)=\{\psi: |\psi|^2 f$ is summable with respect to the Haar measure on $B\}$. Let $H_t$ be the subspace of $L_2(B, f)$ spanned by $\{\chi_g: g<t\}$, where $\chi_g$ denotes the character on $B$ corresponding to the real number $g\in G$. It is clear that $H_t\subseteq H_{t'}$ whenever $t<t'$. It can be shown that either $H_t=H_{t'}$ for all $t$, $t'$ or $\bigcap_t H_t=\{0\}$ and $H_t\subsetneq H_{t'}$ whenever $t<t'$. This has been shown by Helson and Lowdenslager in their paper [3]. The problem of evanescent processes can be stated as follows: Assume that $H_t\neq H_{t'}$ for $t<t'$, then is it always true that $(\bigcap_{t>0} H_t)\ominus H_0\neq\{0\}$?

A well-known result of Helson and Lowdenslager [2] answers the question in the affirmative under the assumption that $\log f$ is summable with respect to the Haar measure on $B$. In what follows we give further evidence in favor of the affirmative answer to the question.

The increasing subspaces $H_t$ give rise to a spectral measure $E$ on the Borel subsets of $R$. For intervals $(a, b]$, $E$ is given by $E(a, b]=$ orthogonal projection on $H_b\ominus H_a$. In $L_2(B, f)$ there is a commutative group $U^g$ of unitary operators defined by $U^g\psi=\chi_g\psi$, $\psi\in L_2(B, f)$, $g\in G$. Further the following two identities are easily verified

(A) $U^g(H_b\ominus H_a)=H_{b+g}\ominus H_{a+g}$ where $a$, $b$ $(a<b)$ are any two real numbers.

(B) For any $\psi\in L_2(B, f)$, $\|E(a, b]\psi-\psi\|^2=\|U^g E(a, b]\psi-U^g\psi\|^2$.

(A) and (B) together imply that $U^g$ and $E$ are connected by the relation $U^g E(\sigma) U^{-g} = E(\sigma + g)$ for all $\sigma \in \mathcal{B}$ and $g \in G$. Helson and Lowdenslager have shown that if $E\{x\} \neq 0$ for some $x$, then the spectral measure $E$ is purely discrete and $E$ has no continuous component. Now it can be shown that $E$ cannot have a component absolutely continuous with respect to the Lebesgue measure on $R$, i.e., there does not exist a nonzero $\psi \in L_2(B, f)$ such that $(E(\cdot)\psi, \psi)$ is absolutely continuous with respect to the Lebesgue measure on $R$. In what follows we show that $E$ has no component absolutely continuous with respect to a nonatomic singular $G$-invariant measure on $R$.

THEOREM 3. *Assume that $E\{x\} = 0$ for all $x$. There does not exist a Borel set $D$ such that*:

(i) *the sets $D+g$, $g \in G$ are mutually disjoint,*

(ii) $E(D) \neq 0$.

PROOF. Suppose not. Then there exists a set $D$ such that the sets $D+g$, $g \in G$, are mutually disjoint and $E(D) \neq 0$. Since $E$ has no discrete spectrum, we can find two nonzero vectors $\Phi$, $\psi$ in $E(D)$ such that $\Phi$ and $\psi$ are mutually orthogonal. Now $U^g\Phi = U^g E(D)\Phi = E(D+g) U^g \Phi \in E(D+g)$ and similarly $U^g\psi \in E(D+g)$. Since the sets $D+g$, $g \in G$, are mutually disjoint, we see that $U^g\Phi \perp \Phi$, $U^g\psi \perp \psi$ for all $g \neq 0$ and $U^g\Phi \perp U^{g'}\psi$ for all $g$, $g'$. So

(i) $(U^g\Phi, \Phi) = \int_B \chi_g(\lambda) |\Phi(\lambda)|^2 f(\lambda) d\sigma = 0$ for $g \neq 0$.

(ii) $(U^g\psi, \psi) = \int_B \chi_g(\lambda) |\psi(\lambda)|^2 f(\lambda) d\sigma = 0$ for $g \neq 0$.

(iii) $(U^g\Phi, \psi) = \int_B \chi_g(\lambda) \Phi(\lambda) \overline{\psi(\lambda)} f(\lambda) d\sigma = 0$ for all $g$.

(Here $d\sigma$ is the normalized Haar measure on $B$.)

The first two equations above say that $|\Phi|^2 f d\sigma$ and $|\psi|^2 f d\sigma$ are nonzero constant multiples of the Haar measure on $B$ and the third equation says that $\Phi\bar{\psi} f$ is equal to zero almost everywhere with respect to the Haar measure on $B$. This is impossible, q.e.d.

4. Let $E$ be a spectral measure on the Borel subsets of $R$ and let $G$ be a countable dense subgroup of $R$. We call a spectral measure $E$ $G$-stationary if there exists a commutative group $U^g$ of unitary operators such that $U^g E(\sigma) U^{-g} = E(\sigma + g)$ for all $\sigma \in \mathcal{B}$ and $g \in G$. If one tries to obtain the canonical representation of $G$-stationary spectral measures like the one there is for a pair of commutative groups of unitary operators satisfying Weyl's commutativity relation one at once faces the following question.

Let $\mu$ be a finite positive measure on $\mathcal{B}$. Call $\mu$ $G$-quasi invariant if $\mu$ and $\mu_g$ are mutually absolutely continuous for all $g \in G$. Here $\mu_g$ is defined by $\mu_g(A) = \mu(A+g)$, $A \in \mathcal{B}$, $g \in G$.

QUESTION 1. $\mu$ is $G$-quasi invariant. Does there exist a $\sigma$-definite measure $m$ on $\mathcal{B}$ such that (i) $m(\sigma+g)=m(\sigma)$ for all $\sigma\in\mathcal{B}$, $g\in G$, (ii) $m$ and $\mu$ are mutually absolutely continuous?

Now suppose that $\mu=\mu^d+\mu^a+\mu^s$ where $\mu^d$ is the atomic part of $\mu$, $\mu^a$ = part of $\mu$ absolutely continuous with respect to the $L$, the Lebesgue measure, and $\mu^s$ = nonatomic singular part of $\mu$. It is easy to see that each component $\mu^d$, $\mu^a$ and $\mu^s$ is separately $G$-quasi invariant.

Further $\mu^a$ and the Lebesgue measure are mutually absolutely continuous. Thus for $\mu^a$ the question raised above has a solution. One can also show easily that the question raised above has a solution for $\mu^d$. Hence in the question raised above one can assume that $\mu$ is nonatomic singular measure.

We give a reformulation of our question in terms of the functions $A(g, \lambda)=(d\mu_g/d\mu)(\lambda)$. One verifies very easily that $A(g, \lambda)$ satisfy the relation $A(g+h, \lambda)=A(g, \lambda)A(h, \lambda+g)$ a.e. $[\mu]$.

THEOREM 4. *Question* 1 *has a solution if and only if there exists a measurable function $B$ such that $A(g, \lambda)=B(\lambda+g)/B(\lambda)$.*

PROOF. Suppose there exists an $m$ as in Question 1. Write $B(\lambda)=(d\mu/dm)(\lambda)$. Then clearly $A(g, \lambda)=(d\mu_g/d\mu)(\lambda)=(d\mu_g/dm)(\lambda)\cdot(dm/d\mu)(\lambda)=(d\mu_g/dm)(\lambda)\cdot 1/B(\lambda)$. Now by the invariance of $m$ under translation by $g$ it is easy to see that $(d\mu_g/dm)(\lambda)=B(\lambda+g)$; thus $A(g,\lambda)=B(\lambda+g)/B(\lambda)$. Conversely suppose that $A(g,\lambda)=B(\lambda+g)/B(\lambda)$ where $B$ is measurable. Define $m$ by $m(\sigma)=\int_\sigma[B(\lambda)]^{-1}d\mu$. It is clear that $m$ and $\mu$ are mutually absolutely continuous. Next to see the $G$-invariance of $m$ we note that

$$\begin{aligned} m(\sigma+g) &= \int_{\sigma+g}[B(\lambda)]^{-1}d\mu = \int_\sigma [B(\lambda+g)]^{-1}d\mu_g(\lambda) \\ &= \int_\sigma [B(\lambda+g)]^{-1}\frac{d\mu_g}{d\mu}(\lambda)d\mu = \int_\sigma \frac{[B(\lambda+g)]^{-1}B(\lambda+g)}{B(\lambda)}d\mu \\ &= \int [B(\lambda)]^{-1}d\mu = m(\sigma), \qquad \text{q.e.d.} \end{aligned}$$

We conclude by making the following remarks.

Assume that $\mu$ of Question 1 is singular. In order that Question 1 have an affirmative solution it is enough that there is a $\mu$-measurable set $D$ such that $D+g$, $g\in G$ are disjoint and $\bigcup_{g\in G}(D+g)$ supports $\mu$. However, there exist singular $G$-quasi invariant measures for which no such $D$ exists. We illustrate this by the following example. Let $C$

be the Cantor ternary set and $\psi$ the Cantor function from $C$ onto $[0, 1]$. $\psi$ is strictly increasing and continuous on $C$ with range $[0, 1]$. Let $P$ be the singular measure on the real line induced by $\psi$. Let $G$ be the group of real members having finitely many terms in their ternary expansions. Let $g_1, g_2, g_3, \cdots$ be a denumeration of $G$. Write $\mu(A) = \sum_{n=1}^{\infty}(1/2^n)P(A+g_n)$, $A \in \mathcal{B}$. Clearly $\mu$ is $G$-quasi invariant. Call two members of $C$ equivalent if their difference belongs to $G$. This equivalence relation partitions $C$. Choose a member from each equivalence class and call the new set $D$. Translates $D+g$, $g \in G$ are disjoint and $\bigcup_{g \in G}(D+g)$ supports $\mu$. But $D$ can never be chosen to be $\mu$-measurable, for the difference of two members of $\psi(D)$ has always finite binary expansions, so that $\psi(D)$ is nonmeasurable. Hence $D = \psi^{-1}(\psi(D))$ is non-$\mu$-measurable.

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Michigan State University and
University of Minnesota