EXTENDING TRIANGULATIONS

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Let Q be a compact piecewise linear (PL) manifold, and M a *proper* compact PL-submanifold of Q. To say M is a proper submanifold means that the boundary $\dot{M} = M \cap \dot{Q}$. A *triangulation* of M will mean a combinatorial manifold K together with a homeomorphism k: $K \to M$ that is compatible with the PL-structure of M; where no confusion can arise k is usually omitted.

DEFINITION. A triangulation J of Q is said to *extend* K if, in the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, dashed, "s"'] & M \arrow[d, phantom, "\cap"] \\ J \arrow[r, "j"'] & Q, \end{tikzcd}$$

the induced map s: $K \to J$ is simplicial.

Recall that the submanifold M is said to be *locally unknotted* in Q if, for some triangulation K of M and extension J of K over Q, the closed star ball pair

$$(\overline{\mathrm{star}}(sv, J),\ s[\overline{\mathrm{star}}(v, K)])$$

is unknotted for each vertex $v \in K$. The choice of K and J is irrelevant, since if this is true for a particular pair K, J, it is true for any subdivisions K', J' and consequently for any other choice. Of course, by [1], local knotting can only occur in codimension 2, and possibly in codimension 1, depending on the validity or otherwise of the PL Schönflies conjecture.

THEOREM. *Every triangulation of M can be extended over Q if and only if M is locally unknotted in Q.*

COROLLARY 1. *Any triangulation of the boundary of a compact PL-manifold can be extended to a triangulation of the whole manifold.*

COROLLARY 2. *If M, instead of being proper in Q, is contained in the interior of Q, and if the codimension is $\geqq 3$, then any triangulation of M can be extended over Q.*

As an example of a nonextendable triangulation in codimension 2, consider the cone on a knotted PL-sphere pair (S^{n+1}, S^{n-1}). This is a

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ball pair (B^{n+2}, B^n) in which B^n is *locally* knotted at the cone point. Triangulate B^n as an n-simplex, and suppose this triangulation can be extended to B^{n+2}. Then the ball pair consisting of the closed star of B^n in this extension, and B^n itself, is unknotted, contradicting the local knottedness of the pair (B^{n+2}, B^n).

Proofs of the theorem and its corollaries will follow a couple of elementary lemmas.

LEMMA 1. *Any triangulation of M has a derived that can be extended over Q.*

PROOF. Let K be a triangulation of M. Since the embedding of M in Q is PL, some subdivision K' of K can be extended to a triangulation J of Q. By [2, Lemma 4], there is a derived $K^{(r)}$ of K that is isomorphic to some subdivision K'' of K'. Finally, by the Corollary to Lemma 3 of [2], there is a subdivision J' of J such that $j\colon J' \to Q$ extends $k\colon K^{(r)} \to M$.

LEMMA 2. *Let (X, Y) and (X_1, Y_1) be two unknotted PL-ball pairs. Then any PL-homeomorphism $h\colon \dot{X}_1 \cup Y_1, Y_1 \to \dot{X} \cup Y, Y$ can be extended to a PL-homeomorphism $\bar{h}\colon X_1, Y_1 \to X, Y$.*

This result occurs as Lemma 18 in [2].

PROOF OF THE THEOREM. Suppose firstly that M is locally unknotted in Q. Given a triangulation K of M there is, by Lemma 1, a derived that can be extended over Q. Now any derived of a finite complex is obtained by a finite number of stellar subdivisions—each such being the result of starring some simplex at an interior point. Therefore (by induction on the number of stellar subdivisions) it is sufficient to prove that if σK is obtained from K by a single stellar subdivision, and if σK can be extended over Q, then K can be extended over Q.

Let σK be obtained by starring the simplex $A \in K$ at the interior point $\hat{A}$, and let J be an extension of σK over Q, i.e. in

$$\sigma K \xrightarrow{k} M \subset Q \xleftarrow{j} J$$

the induced map $s\colon \sigma K \to J$ is simplicial. Some further notation is needed; let L be the subcomplex $s(\sigma K)$ of J, u the vertex $s\hat{A}$, and

$$X = \overline{\text{star}}(u, J), \qquad Y = \overline{\text{star}}(u, L).$$

Then, since M is a proper locally unknotted submanifold of Q, the pair (X, Y) is an unknotted ball pair. It is a cone pair with vertex u, and the idea behind the remainder of this proof is simply to replace this pair by a suitable new cone pair—this replacement will have the

effect of straightening out

$$s[\overline{\text{star}}(A, K)]$$

so that s looks linear.

Choose a vertex v of $s\dot{A}$. Without loss of generality the complex J can be assumed to have the following two properties:

(a) J is contained in a Euclidean space E^n in such a way that all vertices except v lie in a linear subspace E^{n-1} and v lies in $E^n - E^{n-1}$.

(b) $$\overline{\text{star}}(v, J) \cap X = \overline{\text{star}}(v, X).$$

(J is contained in some Euclidean space. Therefore, to accomplish (a), make a small linear move of v into an extra dimension keeping all other vertices of J fixed. If (b) is not satisfied, the remedy is to work throughout with a first derived of J mod L. The latter is equally well an extension of σK over Q, and satisfies (b).) Construct a new complex as follows:

$$J_1 = (J - X) \cup (v * [\dot{X} - \text{star}(v, X)]),$$

where $*$ denotes linear join, and $\text{star}(v, \dot{X})$ the open star of v in $\dot{X}$. Notice the join is well defined by (a), and J_1 a simplicial complex by (b). It remains to produce a suitable homeomorphism j_1: $J_1 \to Q$.

To arrange consistency of notation with Lemma 2, let

$$X_1 = v * [X - \text{star}(v, X)], \qquad Y_1 = v * [Y - \text{star}(v, Y)].$$

Again using local unknottedness, $(\dot{X}, \dot{Y})$ is an unknotted, and therefore locally unknotted, sphere pair. Thus

$$(\overline{\text{star}}(v, X), \overline{\text{star}}(v, Y))$$

is an unknotted ball pair, and so the complementary pair in $(\dot{X}, \dot{Y})$ is also unknotted. Consequently (X_1, Y_1) is exhibited as the cone on an unknotted ball pair, and is therefore itself unknotted. Let

$$f: Y_1 \to \overline{\text{star}}(A, K)$$

be the isomorphism defined as the linear extension of s^{-1} on the vertices. Then the identity and the composition

$$Y_1 \xrightarrow{f} \overline{\text{star}}(A, K) \xrightarrow{s} Y$$

together define a PL-homeomorphism h: $\dot{X}_1 \cup Y_1, Y_1 \to \dot{X} \cup Y, Y$. By Lemma 2, this may be extended to $\bar{h}$: $X_1, Y_1 \to X, Y$. Finally, define j_1: $J_1 \to Q$ by

$$j_1 | J_1 - X_1 = j, \qquad j_1 | X_1 = j\bar{h}.$$

Then by construction $j_1\colon J_1 \to Q$ extends $k\colon K \to M$.

Conversely, suppose M is locally knotted in Q. To complete the proof one needs a triangulation of M that cannot be extended over Q. Let x be a point at which M is locally knotted in Q. Then it is enough to produce a triangulation of M in which

(i) if $x \in \mathring{M}$ (the interior of M) then x lies in the interior of an m-simplex, or

(ii) if $x \in M$, then x lies in the interior of an $(m-1)$-simplex. For let K be such a triangulation, and assume K can be extended to a triangulation J of Q. Let A be the simplex of K that has x in its interior, and let K', J' result from K, J by starring A at x. Then

$$(\mathrm{link}(x, J'), \mathrm{link}(x, K')) = A * (\mathrm{link}(A, J), \mathrm{link}(A, K))$$

and therefore is an unknotted sphere (ball) pair for $x \in \mathring{M}$ $(\dot{M})$, contradicting the local knotting of M in Q at x. Triangulations of the required type can be constructed as follows. Let $k\colon K \to M$ be any triangulation of M. Choose a point y of M such that $k^{-1}y$ lies in the interior of an m-simplex if $x \in \mathring{M}$, and in the interior of an $(m-1)$-simplex of $\dot{K}$ if $x \in \dot{M}$. Using the homogeneity of M there is a PL-homeomorphism $h\colon M \to M$ that sends y to x. Then the triangulation $hk\colon K \to M$ has the required property.

The proof of the theorem is now complete.

Proof of Corollary 1. Let M be the manifold in question. Add a collar to M and denote the resulting manifold by Q. Then $\dot{M}$ is a proper locally unknotted submanifold of Q and so the theorem is applicable. Therefore any triangulation of $\dot{M}$ can be extended to a triangulation of Q, and of course M must appear as a subcomplex.

Proof of Corollary 2. Suppose K is a given triangulation of M, and let N be a relative regular neighbourhood of M mod $\dot{M}$ in $\mathring{Q}$. By [1] M is locally unknotted in N. First apply the theorem to extend K over N, then apply Corollary 1 to the manifold $Q - \mathring{N}$ to complete the extension.

Two questions have been neglected. If M is locally knotted in Q, which triangulations of M are extendable over Q? When is it possible to extend triangulations for polyhedra? Information on both of these will be given in a subsequent paper by E. C. Zeeman [3].

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References

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