# ON THE SPECTRA OF CLASSES OF ALGEBRAS

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1. **Introduction.** Let $\Phi$ be a first-order sentence of a first-order logic with equality. H. Scholz [13] proposed the problem of characterizing the spectrum of $\Phi$, $\mathrm{Sp}(\Phi)$, the set of positive integers $n$ such that $\Phi$ has an $n$-element model. Asser [1] and Mostowski [12] derived some properties of $\mathrm{Sp}(\Phi)$, in particular that $\mathrm{Sp}(\Phi)$ is always an elementary set. T. Evans [4] investigated the special case where $\Phi$ is an identity (considered as a universal sentence) and noticed that then $S=\mathrm{Sp}(\Phi)$ has the following two properties:

(i) $1\in S$;

(ii) $a, b\in S$ implies $ab\in S$ (i.e. $S\cdot S\subseteq S$),

and Evans showed that certain sets of positive integers can be represented as spectra of identities, for instance the set $\{nd; n=1, 2, \cdots\}\cup\{1\}$.

Generalizing Scholz's concept, for any class of algebras $K$, we can define $\mathrm{Sp}(K)$ as the set of positive integers $n$ such that there is an $n$-element algebra in $K$. (To get a proper generalization it would be necessary to consider classes of structures; however this would not change the results.) In this note we take up the characterization problem of spectra for equational and universal classes.

2. **Results and problems.**[2] An equational class $K$ of algebras is the class of all algebras (of a given type) satisfying some fixed set of identities.

THEOREM 1. *The spectrum of an equational class has properties* (i) *and* (ii). *Conversely, if $S$ is a set of positive integers satisfying* (i) *and* (ii), *then there exists an equational class $K$ with* $\mathrm{Sp}(K)=S$, *and $K$ can be chosen to be of finite type, that is there are only finitely many operations.*

However, we cannot, in general, choose $K$ to be an equational class defined by a finite set of identities.

THEOREM 2. *There exists an equational class $K$ such that*

$$\mathrm{Sp}(K) = \mathrm{Sp}(K_1)$$

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[2] The results were announced in Grätzer [9]. In the abstract, Theorem 1 was presented in a weaker form.

*for no equational class $K_1$ defined by a finite set of identities.*

Whatever spectrum we can get with a finite set of identities, we can get with at most two:

THEOREM 3. *Let $K$ be an equational class defined by a finite set of identities. Then there exists an equational class $K_1$ defined by two identities such that*

$$\mathrm{Sp}(K) = \mathrm{Sp}(K_1).$$

REMARK. In the first version of this paper I had "four" rather than "two" in Theorem 3. Professor B. H. Neumann suggested to me that the result of [10] may be used to reduce "four" to "one." I did not quite succeed in it, but, at least, "four" is reduced to "two."[3]

For universal classes (defined by a set of universal sentences) we get the following result:

THEOREM 4. *Let $S$ be an arbitrary set of positive integers. Then there exists a universal class $K$ with* $\mathrm{Sp}(K) = S$.

The problem of characterization of $\mathrm{Sp}(K)$, where $K$ is elementary or equational, defined by one identity remains unsolved.

3. **Preliminaries.** If $K$ is a class of algebras $\boldsymbol{I}(K)$, $\boldsymbol{H}(K)$, $\boldsymbol{S}(K)$, $\boldsymbol{P}(K)$, $\boldsymbol{P}_S(K)$, $\boldsymbol{P}_P(K)$ will denote the classes of all isomorphic copies, homomorphic images, subalgebras, direct products, subdirect products, and prime products of algebras in $K$, respectively. A polynomial $p$ of an algebra $\mathfrak{A} = \langle A; F\rangle$ is a mapping of $A^n$ into $A$ (for some natural number $n$) which can be expressed as a composition of the projections and the operations. A polynomial symbol $\boldsymbol{p}$ is the same as a term in logic; if $\boldsymbol{p}$ is a polynomial symbol and $\mathfrak{A}$ an algebra, then $\boldsymbol{p}^{\mathfrak{A}}$ denotes the polynomial of $\mathfrak{A}$ induced by $\boldsymbol{p}$ in the obvious sense.

The algebra $\mathfrak{A}$ is *primal* if for every natural number $n$, every $n$-ary function on $A$ is a polynomial and $|A| > 1$. A class $K$ of primal algebras is a *primal cluster* (Foster [5]), if for $\{\mathfrak{A}_0, \cdots, \mathfrak{A}_{n-1}\} \subseteq K$, and for any choice of polynomials $p_i$ of $\mathfrak{A}_i$, $i = 0, \cdots, n-1$, there is a polynomial symbol $\boldsymbol{p}$ such that

$$\boldsymbol{p}^{\mathfrak{A}_i} = p_i \quad \text{for } i = 0, \cdots, n-1.$$

4. **Proof of the results.**

PROOF OF THEOREM 1. Properties (i) and (ii) are obvious for $\mathrm{Sp}(K)$

[3] (Added November 8, 1966.) Professor B. H. Neumann kindly sent me the manuscript of his recent paper, *On a problem of G. Grätzer*, submitted for publication to the Publ. Math. (Debrecen), in which "two" is replaced by "one" in our Theorem 3. In fact he proves that the class $K_1$ constructed in the proof of Theorem 3 is an equational class which can be defined by a single identity.

if $K$ is equational. Now let $S$ be given satisfying (i) and (ii). It follows from Foster [5], that there is a primal cluster $K_0$, with $\mathrm{Sp}(K_0)=S$, $K_0=\{\mathfrak{A}_0, \mathfrak{A}_1, \cdots\}$, $\langle A_i;\ F\rangle=\langle A_i;\ 0,\ 1,\ \cdot,\ \frown\rangle$ where $\frown$ is a unary operation which cyclically permutes $A_i$, $0\frown=1$ and $x\cdot 0=0$, $x\cdot 1=x$. Let

$$x^{(0)} = x, \quad x^{(1)} = x\frown, \quad \text{and} \quad x^{(k)} = (x^{(k-1)})\frown.$$

We introduce a ternary operation $f$ on each $\mathfrak{A}_i$ such that if $k\not\equiv t$ (mod $|A_i|$), then $f(0^{(k)}, 0^{(k)}, 0^{(t)})=0$ and $f(0^{(t)}, 0^{(k)}, 0^{(t)})=1$. Let

$$F' = F \cup \{f\}$$

and

$$\mathfrak{A}_i' = \langle A_i; F'\rangle, \qquad K_0' = \{\mathfrak{A}_i' \mid \mathfrak{A}_i \in K_0\}.$$

Obviously, $K_0'$ is a primal cluster.

We claim that $K=\boldsymbol{HSP}(K_0')$ is an equational class with $\mathrm{Sp}(K)=S$. It is known (Birkhoff [2]), that $K$ is an equational class.

Since $K_0'\subseteq \boldsymbol{HSP}(K_0')$, we have $\mathrm{Sp}(K)\supseteq S$. Now let $\mathfrak{A}$ be a finite algebra in $K$. First, let us assume that $\mathfrak{A}\in\boldsymbol{HSP}(L)$ for no finite $L\subseteq K_0'$. By a result of Jónsson [11],

$$\mathfrak{A} \in \boldsymbol{IP}_S\boldsymbol{HSP}_P(K_0'),$$

and so $\mathfrak{A}$ is isomorphic to a subalgebra of a direct product

$$\prod(\mathfrak{B}_\lambda \mid \lambda \in \Lambda),$$

where $\mathfrak{B}_\lambda\in\boldsymbol{HSP}_P(K_0')$, and all the $\mathfrak{B}_\lambda$ are finite. Now we prove that either $|B_\lambda|=1$ or $\mathfrak{B}_\lambda\in\boldsymbol{I}(K_0')$. Indeed, if neither holds for some $\lambda\in\Lambda$, then there is a prime product

$$\prod_{\mathfrak{D}}(\mathfrak{A}_\alpha \mid \alpha \in \Omega), \quad \mathfrak{A}_\alpha \in K_0' \quad \text{for } \alpha \in \Omega,$$

and a subalgebra $\mathfrak{B}$ of this prime product, and a congruence relation $\Theta$ on $\mathfrak{B}$, such that $\mathfrak{B}/\Theta\cong\mathfrak{B}_\lambda$. The elements of $\mathfrak{B}$:

$$0,\ 0^{(1)} = 1,\ 0^{(2)}, \cdots$$

are all distinct, since $I=\{i \mid 0^{(k)}=0^{(t)} \text{ in } \mathfrak{A}_i\}$ is finite if $k\neq t$ and so $\{\alpha \mid \mathfrak{A}_\alpha\cong\mathfrak{A}_i \text{ for some } i\in I\}\notin\mathfrak{D}$ (if it were an element of $\mathfrak{D}$, then a result of [7] would imply that $\prod_{\mathfrak{D}}(\mathfrak{A}_\alpha \mid \alpha\in\Omega)\in\boldsymbol{I}(K_0')$). Since $\mathfrak{B}/\Theta$ is finite we have, for some $k\neq t$, $0^{(k)}\equiv 0^{(t)}(\Theta)$. As above, we can prove that in $\mathfrak{B}$

$$f(0^{(k)}, 0^{(k)}, 0^{(t)}) = 0$$

and

$$f(0^{(t)}, 0^{(k)}, 0^{(t)}) = 1.$$

Thus we get that $0 \equiv 1(\Theta)$ and so for every $a \in B$,

$$a \cdot 0 = 0 \equiv a \cdot 1 = a(\Theta),$$

that is $\mathfrak{B}/\Theta$ has only one element, contrary to $\mathfrak{B}/\Theta \cong \mathfrak{B}_\lambda$ and $|B_\lambda| \neq 1$.

So we proved that if $|A| \neq 1$, then $\mathfrak{A} \in \boldsymbol{IP}_S(K_0')$. Then by a result of C. C. Chang [**3**], $\mathfrak{A} \in \boldsymbol{IP}_S(L)$ for some finite $L \subseteq K_0'$, thus $\mathfrak{A} \in \boldsymbol{HSP}(L)$, contrary to our assumption.

Thus we have proved that $\mathfrak{A} \in \boldsymbol{HSP}(L)$ for some finite $L \subseteq K_0'$. Then by a result of Foster [**6**] (see also [**8**] and [**11**]), $\mathfrak{A}$ is isomorphic to some

$$(\mathfrak{A}_{i_0}')^{m_0} \times \cdots \times (\mathfrak{A}_{i_{k-1}}')^{m_{k-1}}, \qquad \mathfrak{A}_{i_j} \in K_0, \quad j = 0, \cdots, k-1,$$

and by (i) and (ii), $|A| \in S$, completing the proof of Theorem 1.

By Theorem 1, the cardinality of the set of all $\mathrm{Sp}(K)$, for equational classes $K$, is the power of the continuum, while it follows from Asser and Mostowski's result that there are only countably many $\mathrm{Sp}(K)$, for equational classes defined by a finite set of identities. Hence Theorem 2 follows. A "concrete" example is the following: Let $D$ be a nonrecursive set of natural numbers, $\overline{D} = \{p_i \mid i \in D\}$, where $p_i$ is the $i$th prime, and $S$ the set of all positive integers which are of the form

$$g_0^{n_0} \cdots g_{k-1}^{n_{k-1}}, \quad \text{with } g_0, \cdots, g_{k-1} \in \overline{D}.$$

Then $S$ satisfies (i) and (ii) of Theorem 1, so $S = \mathrm{Sp}(K)$ for some equational class $K$. But $S = \mathrm{Sp}(K_1)$ for an equational class $K_1$ defined by a finite set of identities would imply that $S$ is recursive. If $S$ were recursive so would be $\overline{D}$, hence $D$, which is a contradiction.

Proof of Theorem 3. $\langle G; \circ \rangle$ is called a group with right-division $\circ$, if there is a group $\langle G; \cdot, {}^{-1} \rangle$ such that $x \circ y = x \cdot y^{-1}$, or, equivalently, if $\langle G; \cdot, {}^{-1} \rangle$ is a group, where $x \cdot y = x \circ ((x \circ x) \circ y)$ and $y^{-1} = (y \circ y) \circ y$. The identity of the group will be denoted by $1 (= x \circ x)$.

Let $F$ be a set of operations, $\boldsymbol{p}$ a polynomial symbol, and $\circ \in F$. We will use the following result:

Theorem [**10**]. *For the algebra $\mathfrak{A} = \langle A; F \rangle$ the following two conditions are equivalent*:

($\alpha$) *$\langle A; \circ \rangle$ is a group with right-division $\circ$ and $\mathfrak{A}$ satisfies the identity*

$$\boldsymbol{p} = 1; \tag{1}$$

($\beta$) *$\mathfrak{A}$ satisfies the identity*

$$(2) \qquad x_0 \circ \{[(((x_0 \circ x_0) \circ p) \circ x_1) \circ x_2] \circ (((x_0 \circ x_0) \circ x_0) \circ x_2)\} = x_1.$$

This was proved in [10] in the special case when $p$ is expressed in terms of $\circ$. However, this fact is used only once in the proof (lines 29–33 on p. 218). This step can be carried out not using this special assumption as follows (we use here the terminology of [10]): We want to prove that $w=e$; we already know that $w$ is constant. Substitute $x=y=z=e$ in (3.21); using (3.51) we get

$$eewpepepepp = e.$$

Since all left- and right-multiplications are one-to-one, an easy argument shows that $w=e$.

Now we return to the proof of Theorem 3.

Let $K$ be defined by the identities

$$p_i = q_i, \quad i = 0, \cdots, n-1,$$

and let $F$ denote the set of operations of the algebras in $K$. Let $\circ$ and $\vee$ be binary operations which do not occur in $F$ and consider the following identities:

(I) the identity (2) with

$$p = [(\cdots((p_0 \circ q_0) \vee (p_1 \circ q_1)) \vee \cdots) \vee (p_{n-1} \circ q_{n-1})] \circ [(x \vee (y \vee z)) \circ ((y \vee x) \vee (z \vee z))],$$

(II) $x_0 \vee (x_0 \circ x_0) = x_0$,

where $x$, $y$, $z$ are variables which occur in no $p_i$ or $q_i$.

Set $F_1 = F \cup \{\circ, \vee\}$ and let $K_1$ be the equational class defined by (I) and (II). We claim that $S_p(K) = S_p(K_1)$.

Let $m \in \mathrm{Sp}(K)$; then there is an $\mathfrak{A} = \langle A; F\rangle \in K$ with $|A| = m$. Enumerate $A: a_0, \cdots, a_{m-1}$ and define $a_i \vee a_j = a_{\max\{i,j\}}$ and $a_i \circ a_j = a_k$, where $\hat{i} - \hat{j} = \hat{k}$, where $\hat{i}$, $\hat{j}$, $\hat{k}$ is the residue class of the integers modulo $m$, represented by $i$, $j$ and $k$, respectively. Then $\langle A; \vee\rangle$ is a semilattice with $a_0$ as zero, $\langle A; \circ\rangle$ is a group with right-division $\circ$ with $a_0$ as identity. Since $\mathfrak{A}$ satisfies $p_i = q_i$, $\mathfrak{A}_1 = \langle A; F_1\rangle$ satisfies

$$(\cdots((p_0 \circ q_0) \vee (p_1 \circ q_1)) \cdots) \vee (p_{n-1} \circ q_{n-1}) = 1$$

and

$$x \vee (y \vee z) = (y \vee x) \vee (z \vee z),$$

thus $\mathfrak{A}_1$ satisfies (I) and (II). Therefore, $\mathfrak{A}_1 \in K_1$, $m \in S_p(K_1)$.

Conversely, let $m \in S_p(K_1)$; then there is an $\mathfrak{A}_1 = \langle A; F_1\rangle \in K_1$ with $|A| = m$. Then $\mathfrak{A}_1$ satisfies (I) and (II) and by the Theorem [10], $\langle A; \circ\rangle$ is a group with $\circ$ as right-division and $p = 1$ in $\mathfrak{A}_1$, where $p$ is

given in (I). By (II),

$$x_0 \vee 1 = x_0 \tag{3}$$

and so the substitution $x=y=z=1$ in $p=1$ gives

$$(\cdots(p_0 \vee q_0) \vee \cdots) \vee (p_{n-1} \vee q_{n-1}) = 1 \tag{4}$$

and also

$$(x \vee (y \vee z)) \circ ((y \vee x) \vee (z \vee z)) = 1. \tag{5}$$

(5) then yields

$$x \vee (y \vee z) = (y \vee x) \vee (z \vee z). \tag{6}$$

The substitution $z=1$, in (6) and (3) gives

$$z = z \vee z, \tag{7}$$

and the substitution $z=1$, and (3) and (7) give

$$x \vee y = y \vee x. \tag{8}$$

(6), (7) and (8) yield

$$x \vee (y \vee z) = (x \vee y) \vee z. \tag{9}$$

By (3), (7), (8), (9), $\langle A; \vee, 1\rangle$ is a semilattice with 1 as zero element. Since a join is zero, only if all elements are zero, (4) yields

$$p_i \circ q_i = 1, \qquad 0 \leqq i < n,$$

that is

$$p_i = q_i, \qquad 0 \leqq i < n.$$

Thus $\mathfrak{A}=\langle A; F\rangle$ satisfies these identities and so $\mathfrak{A}\in K$, $m\in \mathrm{Sp}(K)$, which was to be proved.

Proof of Theorem 4. We take the class $K_0$ of the proof of Theorem 1 and form $K=\boldsymbol{ISP}_P(K_0)$. Let $\mathfrak{A}\in K$, where $\mathfrak{A}$ is a finite algebra. Let $\Phi$ be a first order sentence which holds for an algebra $\mathfrak{B}$ if and only if it has a subalgebra isomorphic to $\mathfrak{A}$ (such a $\Phi$ exists since there are finitely many operations). Let $\mathfrak{A}$ be isomorphic to a subalgebra of the prime product

$$\mathfrak{B} = \prod\nolimits_{\mathfrak{D}}(\mathfrak{B}_i \mid \mathfrak{B}_i \in K, i \in I).$$

Then $\Phi$ holds in $\mathfrak{B}$, therefore $\Phi$ holds in some $\mathfrak{B}_i\in K$. But a primal algebra has no proper subalgebra, so $|A| = |B_i| \in S$. Thus $\mathrm{Sp}(K)=S$. Since $\boldsymbol{ISP}_P(K_0)$ is known to be a universal class, this completes the proof of Theorem 4.

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