1. Introduction. Ehrenfeucht and Feferman have shown [1] that all recursively enumerable sets $X$ of natural numbers are "representable" in any consistent recursively enumerable theory $S$ in which all recursive functions are definable (in the sense of Tarski-Mostowski-Robinson [4]) and which has a formula $x_1 \leq x_2$ satisfying conditions (i), (ii) below for each natural number $n$:

(i) $\vdash_S x_1 \leq \bar{n} \equiv x_1 = 0 \lor x_1 = 1 \lor \cdots \lor x_1 = n$

(ii) $\vdash_S x_1 \leq \bar{n} \lor \bar{n} \leq x_1$.

(Here $\bar{n}$ is the (closed) numerical term of $S$ corresponding to $n$, i.e. $\Delta_n$ of [4, p. 44].) (By a construction of Cobham (see [3, p. 121] for details), (ii) is redundant in the presence of (i) and the definability in $S$ of the successor function.) That is, for such an $X$, there is a formula $\Phi(x_1)$ of $S$ (with exactly one free variable $x_1$) such that for every $n$,

$$n \in X \iff \vdash_S \Phi(\bar{n}).$$

The argument is to show that there is some creative set $C$ representable in $S$, from which the result follows by the reducibility of $X$ to $C$ by some recursive function (Myhill). Shepherdson has obtained the result [3] more directly by an elegant adaptation of Rosser-type arguments, much as Bernays obtained results of Myhill on theories. In [2] Ritchie and Young show that in every consistent recursively enumerable extension $S$ of R. M. Robinson's system $R$ ([4, pp. 52–53]), all partial recursive functions $\phi$ are "strongly representable." That is, for such a $\phi$, there is a formula $\Phi(x_1, x_2)$ of $S$ such that for all $m, n$,

(iii) $\phi(m) = n \iff \vdash_S \Phi(\bar{m}, \bar{n})$;

and further,

(iv) $\vdash_S (E_1 x_2) \Phi(x_1, x_2)$.

This result not only yields that of Ehrenfeucht and Feferman as an immediate corollary but also gives a neat characterization of the class of partial recursive functions, in addition to showing that the condition (iii) of [4, p. 45] for definability of a total function by a formula $\Phi$ (viz. for each $n$

$$\vdash_S \Phi(\bar{n}, x_2) \land \Phi(\bar{n}, x_3) \supset x_2 = x_3$$

implies the stronger condition obtained by replacing $\bar{n}$ by a variable

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x_1. The argument uses a theorem on exact separation of disjoint recursively enumerable sets due to Putnam and Smullyan; it is interesting to note that Shepherdson \[3\] obtains this separation result with his direct methods.

The present note gives a direct proof of a slight generalization of the theorem of Ritchie and Young alluded to above. Namely, let S be any consistent recursively enumerable theory in which every recursive relation is definable (in the sense of Tarski-Mostowski-Robinson \[4, p. 44\]) and which has a formula x_1 \leq x_2 satisfying (i) above as well as (ii)' below:

(ii)' \vdash_S x_1 \leq x_2 \vee x_2 \leq x_1.

Alternatively, we may assume that S satisfies just (i) and that every recursive function is definable in S (cf. \[3, p. 121\]). In either case we have the following.

**Theorem.** Every partial recursive function is strongly representable in S.

That is, if \( \phi \) is a partial recursive function then there is a formula \( \Phi(x_1, x_2) \) of S such that for all \( m, n \), (iii) above holds as well as (iv).

2. **Weak representability.** We say that the partial function \( \phi \) is weakly represented in S by \( \Phi(x_1, x_2) \) provided that (iii) above holds as well as

(iv) \[ (iv) \quad \vdash_S \Phi(x_1, x_2) \wedge \Phi(x_1, x_3) \supseteq x_2 = x_3. \]

**Theorem 1.** Every partial recursive function \( \phi \) is weakly representable in S.

In what follows let \( \phi \) be a fixed partial recursive function. Then according to the Enumeration Theorem of Kleene there is a recursive predicate \( T(u, w) \) and a recursive function \( U \) such that

\[
\phi(u) = v \iff (\exists w)[T(u, w) \& U(w) = v].
\]

(We may take \( T(u, w) \) as the \( T_1(e, u, w) \) of \[IM, p. 330\], and the \( U \) found there, where \( e \) is any index for \( \phi \).) We may assume that in particular \( T \) has the property that \( T(u, w_1) \& T(u, w_2) \Rightarrow w_1 = w_2 \). Let the recursive relations \( T(u, w) \) and \( U(w) = v \) be defined in S by \( \exists(x_1, x_2) \) and \( \forall(x_3, x_2) \), respectively, and let \( \exists(x, y, z) \) be the conjunction of the two formulas

\[
\exists(x, z) \wedge (z_1) [z_1 \leq z \supset [\exists(x, z_1) \supset z_1 = z]]
\]

and

\[
\forall(z, y) \wedge (y_1) [y_1 \leq y \supset [\forall(z, y_1) \supset y_1 = y]].
\]
Lemma 1. For any \( u, v, w, \)

\[ T(u, w) \& U(w) = v \iff \vdash_S \mathcal{D}(u, \bar{v}, \bar{w}). \]

Proof. It suffices to show the implication to the right, so suppose that \( T(u, w) \) and \( U(w) = v \). We have then \( \vdash_S \mathcal{J}(u, \bar{w}) \land \mathcal{U}(\bar{w}, \bar{v}) \) by the choice of \( \mathcal{J} \) and \( \mathcal{U} \). But also for every \( n \leq w \) we have

\[ \vdash_S [\mathcal{J}(u, \bar{n}) \supset n = \bar{w}] \]

(since the propositional components of this, for each fixed \( n \), can be proved or disproved appropriately), so by (i) we have

\[ \vdash_S (z_1 \leq \bar{w} \supset [\mathcal{J}(u, z_1) \supset z_1 = \bar{w}]). \]

In the same way we have

\[ \vdash_S (y_1 \leq \bar{v} \supset [\mathcal{U}(\bar{w}, y_1) \supset y_1 = \bar{v}]) \]

and thus \( \vdash_S \mathcal{D}(u, \bar{v}, \bar{w}) \).

Lemma 2. \( \vdash_S \mathcal{D}(x, y_2, z_2) \land \mathcal{D}(x, y_3, z_3) \supset y_2 = y_3. \)

Proof. From (ii)' we have \( \vdash_S z_2 \leq z_3 \lor z_3 \leq z_2 \). Thus by specializing the variable \( z_1 \) in \( \mathcal{D}(x, y_2, z_2) \) to \( z_3 \), and in \( \mathcal{D}(x, y_3, z_3) \) to \( z_2 \), we obtain

\[ \vdash_S \mathcal{D}(x, y_2, z_2) \land \mathcal{D}(x, y_3, z_3) \supset z_2 = z_3. \]

From this equality and by a similar manipulation of the final clauses of \( \mathcal{D}(x, y_2, z_2) \) and \( \mathcal{D}(x, y_3, z_3) \) we obtain the desired result.

Proof of Theorem 1. Let \( Q, P \) be recursive predicates defined as follows:

\[ Q(u, v, w, q) \iff q \text{ is (the G"odel number of) a proof (in } S) \text{ of } \mathcal{D}(u, \bar{v}, \bar{w}) \]

\[ P(u, v, r, p) \iff p \text{ is (the G"odel number of) a proof (in } S) \text{ of } A_r^{(3)}(u, \bar{v}, \bar{r}) \]

where \( A_r^{(3)}(x_1, x_2, x_3) \) is the formula of \( S \) whose G"odel number is \( r \) and which contains no variables free other than \( x_1, x_2, x_3 \). Then \( P, Q \) are represented in \( S \) by some \( \mathcal{P}, \mathcal{Q} \), respectively. Let \( r_0 \) be the G"odel number of the formula

\[ (Ex) \mathcal{D}(x_1, x_2, z) \land (x_4)[\mathcal{Q}(x_1, x_2, x_3, x_4) \supset (Ex_6)[x_6 \leq x_4 \land (Ex_6)[x_6 \leq x_5 \land Q(x_1, x_2, x_8, x_8)]]. \]

We claim that \( \phi \) is weakly represented by \( \Phi \), where \( \Phi(x_1, x_2) \) is the formula \( A_r^{(3)}(x_1, x_2, \bar{r}_0) \).

Now suppose that \( \vdash_S \mathcal{D}(u, \bar{v}) \) for some \( u, v \). Let \( p \) be (the number of) a proof; then \( \vdash_S \mathcal{P}(u, \bar{v}, \bar{r}_0, \bar{p}) \) according to the definition of \( \mathcal{P} \) by \( \mathcal{Q} \) and our supposition. Hence by specialization of \( x_4 \) to \( \bar{p} \) in the definition of \( \Phi \) we have (upon application of modus ponens) that
\begin{align*} \vdash_S (Ex_b)[x_6 \leq \bar{p} \land (Ex_b)[x_6 \leq x_6 \land Q(\bar{u}, \bar{v}, x_6, x_b)]]. \end{align*}

Hence (by a similar argument to the one above in connection with $\mathcal{D}$), $Q(u, v, w, q)$ for some $w, q$ with $w \leq q \leq p$, since $Q$ is defined by $\mathcal{Q}$. Hence (by the meaning of $Q$) there is a proof (in fact, with number $q$) of $\mathcal{D}(\bar{u}, \bar{v}, \bar{w})$. We conclude both $T(u, w)$ and $U(w) = v$, and then that $\phi(u) = v$.

Conversely, let $\phi(u) = v$; take $w$ minimal so that $T(u, w)$ and $U(w) = v$. Then for some minimal $q \geq w$, $q$ is a proof of $\mathcal{D}(\bar{u}, \bar{v}, \bar{w})$ by Lemma 1. Note in passing that $\vdash_S (Exz)Q(u, v, z)$ (by extension of the proof $q$). Now for $p < q$, we have $\vdash_S \neg Q(\bar{u}, \bar{v}, \bar{r}_0, \bar{p})$, since otherwise entails (as above) the existence of $w_1, q_1, p_1$ with $w_1 \leq q_1 \leq p_1 < q$ where $q_1$ is a proof of $\mathcal{D}(\bar{u}, \bar{v}, \bar{w}_1)$; but then $T(u, w_1)$ implies $w_1 = w$, so $q \leq q_1$ (contradicting the choice of $q$). Hence we can show by (i) and (ii) (or (ii)'$)$ that $\vdash_S (x_4)[Q(\bar{u}, \bar{v}, \bar{r}_0, x_4) \supset q \leq x_4]$. But then

\begin{align*} \vdash_S (x_4)[Q(\bar{u}, \bar{v}, \bar{r}_0, x_4) \supset q \leq x_4 \land \bar{w} \leq \bar{q} \land Q(\bar{u}, \bar{v}, \bar{w}, \bar{q})], \end{align*}

so we conclude (by existential quantifications), with the help of the previously noted fact that $\vdash_S (Exz)Q(u, v, z)$, that $\vdash_S \Phi(\bar{u}, \bar{v})$.

All that remains to be shown is (iva), but this follows immediately from Lemma 2.

3. Strong representability. Now we show how to construct another Rosser-type argument to obtain the strong representability of $\phi$. To this end recall the definition of $A_3^{(3)}(x_1, x_2, x_3)$ given above. Define recursive $M, N$ as follows:

$M(u, v, r, p) \iff \bar{p}$ is (the number of) a proof (in $S$) of $\neg \Phi(\bar{u}, \bar{v}) \supset A_3^{(3)}(\bar{u}, \bar{v}, \bar{r})$,

$N(u, v, r, q) \iff q$ is a proof of $\neg \Phi(\bar{u}, \bar{v}) \supset A_3^{(3)}(\bar{u}, \bar{v}, \bar{r})$.

Let $M, N$ be defined in $S$ by $\mathfrak{M}, \mathfrak{N}$, respectively, and let $r_0$ be the Gödel number of the formula

\begin{align*} (x_4)[\mathfrak{M}(x_1, x_2, x_3, x_4) \supset (Ex_b)[x_3 \leq x_2 \land \mathfrak{M}(x_1, x_2, x_3, x_b)]]. \end{align*}

Take $R_0$ as $A_3^{(3)}(x_1, x_2, \bar{r}_0)$ and $\Phi^*(x_1, x_2)$ as the formula

$\Phi(x_1, x_2) \lor [(\exists z) \sim \Phi(x_1, z) \land [(x_2 = \bar{1} \land R_0(x_1, x_2)) \lor (x_2 = \bar{1} \land \sim R_0(x_1, x_2))]]$.

**Theorem 2.** $\Phi^*$ strongly represents $\phi$ in $S$.

**Proof.** By Theorem 1 we infer that if $\phi(u) = v$, then $\vdash_S \Phi(\bar{u}, \bar{v})$, so also $\vdash_S \Phi^*(\bar{u}, \bar{v})$. With the help of the same theorem $\vdash_S \Phi^*(x, y) \land \Phi^*(x, z) \supset y = z$. It is straightforward to verify that $\vdash_S (Ey)\Phi^*(x, y)$, merely using the logical form of $\Phi^*$, so it remains to be shown that if
\( \Phi^*(\bar{u}, \bar{v}) \) is provable in \( S \) for some \( u, v \), then in fact \( \phi(u) = v \). Thus suppose that \( \vdash S \Phi^*(\bar{u}, \bar{v}) \); it suffices to show that \( \vdash S \Phi(\bar{u}, \bar{v}) \) also. First note that if \( v > 1 \), then \( \vdash S \bar{v} \neq 0 \land \bar{v} \neq 1 \) by our assumptions about \( S \). But for such \( v \) it easily follows from the form of \( \Phi^*(\bar{u}, \bar{v}) \) that 
\( \vdash S \Phi(\bar{u}, \bar{v}) \). Hence we need consider only the cases \( v = 0 \) and \( v = 1 \).

**Case** \( v = 0 \). In this case we see that \( \vdash S \sim \Phi(\bar{u}, 0) \supseteq R_0(\bar{u}, 0) \); let \( p \) be the number of a proof. Note that \( M(u, 0, r_0, p) \) holds then, so that 
\( \vdash S \exists \alpha(\bar{u}, 0, r_0, p) \) since \( \exists \alpha \) defines \( M \). Now extend the proof \( p \) by specializing the \( x_4 \) in the definition of \( R_0 \) to \( p \), to obtain

\[
(0^*) \quad \vdash S \sim \Phi(\bar{u}, 0) \supseteq \left( Ex_5 \right) [x_5 \leq \bar{p} \land \exists \alpha(\bar{u}, 0, r_0, x_5)].
\]

Now two possibilities arise. **First**, that \( \vdash S \sim \Phi(\bar{u}, 0) \supseteq \sim R_0(\bar{u}, 0) \). In this subcase, clearly \( \vdash S \Phi(\bar{u}, 0) \) as desired. **Otherwise**, it is not the case that 
\( \vdash S \sim \Phi(\bar{u}, 0) \supseteq \sim R_0(\bar{u}, 0) \), so \( N(u, 0, r_0, q) \) is false for all \( q \), and in particular for \( q \leq p \). Hence for such \( q \), we have \( \vdash S \sim \exists \alpha(\bar{u}, 0, r_0, q) \), since \( \exists \alpha \) defines \( N \), and thus \( \vdash S (x_5) [x_5 \leq q \supseteq \exists \alpha(\bar{u}, 0, r_0, x_5)] \). From this and \((0^*)\) we conclude that \( \vdash S \Phi(\bar{u}, 0) \).

**Case** \( v = 1 \). Now we see that \( \vdash S \sim \Phi(\bar{u}, 1) \supseteq \sim R_0(\bar{u}, 1) \); if \( q \) is the number of a proof, \( N(u, 1, r_0, q) \) holds, so \( \vdash S \exists \alpha(\bar{u}, 1, r_0, q) \). Again it is possible that \( \vdash S \sim \Phi(\bar{u}, 1) \supseteq R_0(\bar{u}, 1) \); and, if so, then \( \vdash S \Phi(\bar{u}, 1) \) as desired. Otherwise, there is no proof in \( S \) of \( \sim \Phi(\bar{u}, 1) \supseteq R_0(\bar{u}, 1) \), and so \( M(u, 1, r_0, p) \) fails for all \( p \). In particular, for \( p \leq q \) we have 
\( \vdash S \exists \alpha(\bar{u}, 1, r_0, p) \), and so (by \((i)\)) \( \vdash S \exists \alpha(\bar{u}, 1, r_0, x_4) \supseteq (x_4 \leq q) \). Now from a proof of \( \exists \alpha(\bar{u}, 1, r_0, q) \) we can construct one of \( q \leq x_4 \supseteq \exists \alpha(\bar{u}, 1, r_0, q) \), so \( \vdash S \exists \alpha(\bar{u}, 1, r_0, q) \supseteq (Ex_5) [x_5 \leq x_4 \supseteq \exists \alpha(\bar{u}, 1, r_0, x_5)] \). By \((ii)\) we have \( \vdash S x_4 \leq q \supseteq x_4 \); so by combining this with the above results,

\[
(1^*) \quad \vdash S \exists \alpha(\bar{u}, 1, r_0 x_4) \supseteq (Ex_5) [x_5 \leq x_4 \supseteq \exists \alpha(\bar{u}, 1, r_0, x_5)].
\]

Generalize on \( x_4 \) in \((1^*)\) to obtain \( \vdash S R_0(\bar{u}, 1) \), and conclude finally that \( \vdash S \Phi(\bar{u}, 1) \) in this case also.

**Bibliography**


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