1. Introduction. Ehrenfeucht and Feferman have shown [1] that all recursively enumerable sets $X$ of natural numbers are “representable” in any consistent recursively enumerable theory $S$ in which all recursive functions are definable (in the sense of Tarski-Mostowski-Robinson [4]) and which has a formula $x_1 \leq x_2$ satisfying conditions (i), (ii) below for each natural number $n$:

(i) $\forall x_1 \leq \bar{n} = x_1 = 0 \lor x_1 = 1 \lor \cdots \lor x_1 = \bar{n},$

(ii) $\forall x_1 \leq \bar{n} \lor \bar{n} \leq x_1.$

(Here $\bar{n}$ is the (closed) numerical term of $S$ corresponding to $n$, i.e. $\Delta_n$ of [4, p. 44].) (By a construction of Cobham (see [3, p. 121] for details), (ii) is redundant in the presence of (i) and the definability in $S$ of the successor function.) That is, for such an $X$, there is a formula $\Phi(x_1)$ of $S$ (with exactly one free variable $x_1$) such that for every $n$,

$$n \in X \iff \forall \neg \Phi(\bar{n}).$$

The argument is to show that there is some creative set $C$ representable in $S$, from which the result follows by the reducibility of $X$ to $C$ by some recursive function (Myhill). Shepherdson has obtained the result [3] more directly by an elegant adaptation of Rosser-type arguments, much as Bernays obtained results of Myhill on theories. In [2] Ritchie and Young show that in every consistent recursively enumerable extension $S$ of R. M. Robinson’s system $R$ ([4, pp. 52–53]), all partial recursive functions $\phi$ are “strongly representable.” That is, for such a $\phi$, there is a formula $\Phi(x_1, x_2)$ of $S$ such that for all $m, n$,

(iii) $\phi(m) = n \iff \forall \neg \Phi(\bar{m}, \bar{n});$

and further,

(iv) $\forall \neg (E_1 x_2) \Phi(x_1, x_2).$

This result not only yields that of Ehrenfeucht and Feferman as an immediate corollary but also gives a neat characterization of the class of partial recursive functions, in addition to showing that the condition (iii) of [4, p. 45] for definability of a total function by a formula $\Phi$ (viz. for each $n$

$$\forall \neg \Phi(\bar{n}, x_2) \land \Phi(\bar{n}, x_3) \lor x_2 = x_3$$

implies the stronger condition obtained by replacing $\bar{n}$ by a variable

Presented to the Society, January 1, 1966; received by the editors June 29, 1966.
The argument uses a theorem on exact separation of disjoint recursively enumerable sets due to Putnam and Smullyan; it is interesting to note that Shepherdson [3] obtains this separation result with his direct methods.

The present note gives a direct proof of a slight generalization of the theorem of Ritchie and Young alluded to above. Namely, let $S$ be any consistent recursively enumerable theory in which every recursive relation is definable (in the sense of Tarski-Mostowski-Robinson [4, p. 44]) and which has a formula $x_1 \leq x_2$ satisfying (i) above as well as (ii)' below:

(ii)' $\vdash_S x_1 \leq x_2 \lor x_2 \leq x_1$.

Alternatively, we may assume that $S$ satisfies just (i) and that every recursive function is definable in $S$ (cf. [3, p. 121]). In either case we have the following.

**Theorem.** Every partial recursive function is strongly representable in $S$.

That is, if $\phi$ is a partial recursive function then there is a formula $\Phi(x_1, x_2)$ of $S$ such that for all $m, n$, (iii) above holds as well as (iv).

2. **Weak representability.** We say that the partial function $\phi$ is *weakly represented* in $S$ by $\Phi(x_1, x_2)$ provided that (iii) above holds as well as

(iv) $\vdash_S \Phi(x_1, x_2) \land \Phi(x_1, x_3) \supseteq x_2 = x_3$.

**Theorem 1.** Every partial recursive function $\phi$ is weakly representable in $S$.

In what follows let $\phi$ be a fixed partial recursive function. Then according to the Enumeration Theorem of Kleene there is a recursive predicate $T(u, w)$ and a recursive function $U$ such that

$$\phi(u) = v \iff (\exists w)[T(u, w) \land U(w) = v].$$

(We may take $T(u, w)$ as the $T_1(e, u, w)$ of [IM, p. 330], and the $U$ found there, where $e$ is any index for $\phi$.) We may assume that in particular $T$ has the property that $T(u, w_1) \land T(u, w_2) \implies w_1 = w_2$. Let the recursive relations $T(u, w)$ and $U(w) = v$ be defined in $S$ by $3(x_1, x_2)$ and $\Phi(x_3, x_2)$, respectively, and let $\Sigma(x, y, z)$ be the conjunction of the two formulas

$$3(x, z) \land (s_1) [s_1 \leq z \supset [3(x, s_1) \supset s_1 = z]]$$

and

$$\Phi(z, y) \land (y_1) [y_1 \leq y \supset [\Phi(z, y_1) \supset y_1 = y]].$$
Lemma 1. For any $u, v, w$,

$$T(u, w) \& U(w) = v \iff \vdash_s D(u, \bar{v}, \bar{w}).$$

Proof. It suffices to show the implication to the right, so suppose that $T(u, w)$ and $U(w) = v$. We have then $\vdash_s \exists(u, \bar{w}) \wedge \forall(\bar{w}, \bar{v})$ by the choice of $\exists$ and $\forall$. But also for every $n \leq w$ we have

$$\vdash_s [\exists(u, \bar{n}) \supset \bar{n} = \bar{w}]$$

(since the propositional components of this, for each fixed $n$, can be proved or disproved appropriately), so by (i) we have

$$\vdash_s (z_1) \leq \bar{w} \supset [\exists(u, z_1) \supset z_1 = \bar{w}].$$

In the same way we have

$$\vdash_s (y_1) \leq \bar{v} \supset [\forall(\bar{w}, y_1) \supset y_1 = \bar{v}]$$

and thus $\vdash_s D(u, \bar{v}, \bar{w}).$

Lemma 2. $\vdash_s D(x, y_2, z_2) \wedge D(x, y_3, z_3) \supset y_2 = y_3.$

Proof. From (ii)' we have $\vdash_s z_2 \leq z_3 \vee z_3 \leq z_2.$ Thus by specializing the variable $z_1$ in $D(x, y_2, z_2)$ to $z_2$, and in $D(x, y_3, z_3)$ to $z_2$, we obtain $\vdash_s D(x, y_2, z_2) \wedge D(x, y_3, z_3) \supset z_2 = z_3.$ From this equality and by a similar manipulation of the final clauses of $D(x, y_2, z_2)$ and $D(x, y_3, z_3)$ we obtain the desired result.

Proof of Theorem 1. Let $Q, P$ be recursive predicates defined as follows:

$Q(u, v, w, q) \iff q$ is (the Gödel number of) a proof (in $S$) of $D(u, \bar{v}, \bar{w})$

$P(u, v, r, p) \iff p$ is (the Gödel number of) a proof (in $S$) of $A^{(3)}(u, \bar{v}, \bar{r})$

where $A^{(3)}(x_1, x_2, x_3)$ is the formula of $S$ whose Gödel number is $r$ and which contains no variables free other than $x_1, x_2, x_3$. Then $P, Q$ are represented in $S$ by some $P, Q$, respectively. Let $r_0$ be the Gödel number of the formula

$$(Ez) D(x_1, x_2, z) \wedge (x_4) [P(x_1, x_2, x_3, x_4) \supset (Ex_5)[z_5 \leq x_4 \wedge (Ex_6)[z_6 \leq x_5 \wedge Q(x_1, x_2, x_6, x_6)]]].$$

We claim that $\phi$ is weakly represented by $\Phi$, where $\Phi(x_1, x_2)$ is the formula $A^{(3)}(x_1, x_2, r_0)$.

Now suppose that $\vdash_s P(u, \bar{v})$ for some $u, v$. Let $p$ be (the number of) a proof; then $\vdash_s P(u, \bar{v}, r, p)$ according to the definition of $P$ by $\Phi$ and our supposition. Hence by specialization of $x_4$ to $p$ in the definition of $\Phi$ we have (upon application of modus ponens) that
\[ \vdash_s (Ex_b) [x_6 \leq \bar{p} \land (Ex_b) [x_6 \leq x_6 \land Q(\bar{u}, \bar{v}, x_6, x_6)]]. \]

Hence (by a similar argument to the one above in connection with \(D\)), \(Q(u, v, w, q)\) for some \(w, q\) with \(w \leq q \leq p\), since \(Q\) is defined by \(Q\). Hence (by the meaning of \(Q\)) there is a proof (in fact, with number \(q\)) of \(D(\bar{u}, \bar{v}, \bar{w})\). We conclude both \(T(u, w)\) and \(U(w) = v\), and then that \(\phi(u) = v\).

Conversely, let \(\phi(u) = v\); take \(w\) minimal so that \(T(u, w)\) and \(U(w) = v\). Then for some minimal \(q \geq w\), \(q\) is a proof of \(D(\bar{u}, \bar{v}, \bar{w})\) by Lemma 1. Note in passing that \(\vdash_s (Ex_z) Q(u, v, z)\) (by extension of the proof \(q\)). Now for \(p < q\), we have \(\vdash_s \neg \phi(\bar{u}, \bar{v}, \bar{r}_0, \bar{p})\), since otherwise entails (as above) the existence of \(w_1, q_1, p_1\) with \(w_1 \leq q_1 \leq p_1 < q\) where \(q_1\) is a proof of \(D(\bar{u}, \bar{v}, \bar{w}_1)\); but then \(T(u, w_1)\) implies \(w_1 = w\), so \(q \leq q_1\) (contradicting the choice of \(q\)). Hence we can show by (i) and (ii) (or (ii)') that \(\vdash_s (x_4) [\phi(\bar{u}, \bar{v}, \bar{r}_0, x_4) \supset q \leq x_4]\). But then

\[ \vdash_s (x_4) [\phi(\bar{u}, \bar{v}, \bar{r}_0, x_4) \supset q \leq x_4 \land \bar{q} \leq q \land Q(\bar{u}, \bar{v}, \bar{w}, q)], \]

so we conclude (by existential quantifications), with the help of the previously noted fact that \(\vdash_s (Ex_z) D(\bar{u}, \bar{v}, z)\), that \(\vdash_s \Phi(\bar{u}, \bar{v})\).

All that remains to be shown is (iva), but this follows immediately from Lemma 2.

3. Strong representability. Now we show how to construct another Rosser-type argument to obtain the strong representability of \(\phi\). To this end recall the definition of \(A_r^{(3)}(x_1, x_2, x_3)\) given above. Define recursive \(M, N\) as follows:

\[ M(u, v, r, p) \iff p \text{ is (the number of) a proof (in } S\text{) of } \sim \Phi(\bar{u}, \bar{v}) \supset A_r^{(3)}(\bar{u}, \bar{v}, \bar{r}), \]

\[ N(u, v, r, q) \iff q \text{ is a proof of } \sim \Phi(\bar{u}, \bar{v}) \supset A_r^{(3)}(\bar{u}, \bar{v}, \bar{r}). \]

Let \(M, N\) be defined in \(S\) by \(\forall \tau, \exists \tau\), respectively, and let \(r_0\) be the Gödel number of the formula

\[ (x_4) [\forall \tau(x_1, x_2, x_3, x_4) \supset (Ex_b) [x_3 \leq x_2 \land \tau(x_1, x_2, x_3, x_6)]]. \]

Take \(R_0\) as \(A_r^{(3)}(x_1, x_2, r_0)\) and \(\Phi^*(x_1, x_2)\) as the formula

\[ \Phi(x_1, x_2) \lor [(z) \sim \Phi(x_1, z) \land [(x_2 = \bar{0} \land R_0(x_1, x_2)) \lor (x_2 = \bar{1} \land \sim R_0(x_1, x_2))]. \]

Theorem 2. \(\Phi^*\) strongly represents \(\phi\) in \(S\).

Proof. By Theorem 1 we infer that if \(\phi(u) = v\), then \(\vdash_s \Phi^*(\bar{u}, \bar{v})\), so also \(\vdash_s \Phi^*(\bar{u}, \bar{v})\). With the help of the same theorem \(\vdash_s \Phi^*(x, y) \land \Phi^*(x, z) \supset y = z\). It is straightforward to verify that \(\vdash_s (Ex_y) \Phi^*(x, y)\), merely using the logical form of \(\Phi^*\), so it remains to be shown that if
\( \Phi^*(\bar{u}, \bar{v}) \) is provable in \( S \) for some \( u, v \), then in fact \( \phi(u) = v \). Thus suppose that \( \vdash_S \Phi^*(\bar{u}, \bar{v}) \); it suffices to show that \( \vdash_S \Phi(\bar{u}, \bar{v}) \) also. First note that if \( v > 1 \), then \( \vdash \bar{v} \neq 0 \lor \bar{v} \neq 1 \) by our assumptions about \( S \). But for such \( v \) it easily follows from the form of \( \Phi^*(\bar{u}, \bar{v}) \) that
\( \vdash_S \Phi(\bar{u}, \bar{v}) \). Hence we need consider only the cases \( v = 0 \) and \( v = 1 \).

Case \( v = 0 \). In this case we see that \( \vdash_S \Phi(\bar{u}, \bar{0}) \supset R_0(\bar{u}, \bar{0}) \); let \( p \) be the number of a proof. Note that \( M(u, 0, r_0, p) \) holds then, so that \( \vdash_S \exists N(\bar{u}, \bar{0}, r_0, p) \) since \( \exists \) defines \( M \). Now extend the proof \( p \) by specializing the \( x_4 \) in the definition of \( R_0 \) to \( \bar{p} \), to obtain

\[
\vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset (Ex_b)[x_b \leq \bar{p} \land \exists(\bar{u}, \bar{0}, r_0, x_b)].
\]

Now two possibilities arise. First, that \( \vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \). In this subcase, clearly \( \vdash_S \Phi(\bar{u}, \bar{0}) \) as desired. Otherwise, it is not the case that \( \vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \), so \( N(u, 0, r_0, q) \) is false for all \( q \), and in particular for \( q \leq p \). Hence for such \( q \), we have \( \vdash_S \sim \exists N(\bar{u}, \bar{0}, r_0, q) \), since \( \exists \) defines \( N \), and thus \( \vdash_S (x_b)\left[x_b \leq \bar{p} \lor \exists(\bar{u}, \bar{0}, r_0, x_b)\right] \). From this and \((0*)\) we conclude that \( \vdash_S \Phi(\bar{u}, \bar{0}) \).

Case \( v = 1 \). Now we see that \( \vdash_S \sim \Phi(\bar{u}, \bar{1}) \supset \sim R_0(\bar{u}, \bar{1}) \); if \( q \) is the number of a proof, \( N(u, 1, r_0, q) \) holds, so \( \vdash_S \exists N(\bar{u}, \bar{1}, r_0, q) \). Again it is possible that \( \vdash_S \sim \Phi(\bar{u}, \bar{1}) \supset R_0(\bar{u}, \bar{1}) \); and, if so, then \( \vdash_S \Phi(\bar{u}, \bar{1}) \) as desired. Otherwise, there is no proof in \( S \) of \( \sim \Phi(\bar{u}, \bar{1}) \supset R_0(\bar{u}, \bar{1}) \), and so \( M(u, 1, r_0, p) \) fails for all \( p \). In particular, for \( p \leq q \) we have \( \vdash_S \sim \exists N(\bar{u}, \bar{1}, r_0, q) \), and so (by (i)) \( \vdash_S \exists N(\bar{u}, \bar{1}, r_0, x_4) \supset (x_4 \leq \bar{q}) \). Now from a proof of \( \exists N(\bar{u}, \bar{1}, r_0, q) \) we can construct one of \( \bar{q} \leq x_4 \supset \bar{q} \leq x_4 \supset \exists(\bar{u}, \bar{1}, r_0, q) \), \( \vdash_S \exists \bar{q} \leq x_4 \supset (Ex_b)[x_b \leq x_4 \land \exists(\bar{u}, \bar{1}, r_0, x_b)] \). By (ii) we have \( \vdash_S x_4 \leq \bar{q} \lor \bar{q} \leq x_4 \); so by combining this with the above results,

\[
\vdash_S \exists N(\bar{u}, \bar{1}, r_0, x_4) \supset (Ex_b)[x_b \leq x_4 \land \exists(\bar{u}, \bar{1}, r_0, x_b)].
\]

Generalize on \( x_4 \) in \((1*)\) to obtain \( \vdash_S \exists R_0(\bar{u}, \bar{1}) \), and conclude finally that \( \vdash_S \Phi(\bar{u}, \bar{1}) \) in this case also.

Bibliography


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