NONUNIQUENESS OF EXTREMAL KERNELS

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Given a "kernel" \( k = k(e^{i\theta}) \in L^q \), then

\[
\Phi(f) = \frac{1}{2\pi i} \int_{|z|=1} f(z)k(z)dz
\]

is a bounded linear functional on \( H^p \) \((1/p + 1/q = 1)\) with norm \( ||\Phi|| \leq ||k||_q \). Two kernels induce the same linear functional, \( k_1 \sim k_2 \), if and only if they differ by an \( H^q \) function. It can be shown that the duality relation, \( ||\Phi|| = \inf \{ ||k_1||_q : k_1 \sim k_2 \} \) holds for \( 1 \leq p \leq \infty \). One may ask whether an extremal kernel exists, i.e., a kernel \( k_1 \sim k \) such that \( ||\Phi|| = ||k_1||_q \), and if so, whether it is unique.

By functional analytic methods, Havinson [1] and Rogosinski and Shapiro [2] showed that an extremal kernel always exists for \( 1 \leq p \leq \infty \) and is unique for \( 1 < p \leq \infty \). However, Rogosinski and Shapiro constructed a counterexample to show that it need not be unique for \( p = 1 \). As their example to show nonuniqueness for \( p = 1 \) was rather complicated, we present a simplification of their example.

Let

\[
k(e^{i\theta}) =
\begin{cases}
1, & 0 \leq \theta < \pi/2, \\
-1, & \pi/2 \leq \theta < \pi, \\
0, & \pi \leq \theta < 2\pi.
\end{cases}
\]

Rogosinski and Shapiro showed that this kernel induces a functional \( \Phi \) on \( H^1 \) with \( ||\Phi|| = 1 \), and thus \( k \) is an extremal kernel. To show this extremal kernel is not unique, it suffices to produce an \( H^\infty \) function \( h \neq 0 \) for which \( ||k + h|| = 1 \).

Let \( h \) be the conformal mapping sending the unit disk to the upper half-disk \( |w| < 1 \), \( \text{Im } w > 0 \), with \( h(1) = -1 \), \( h(i) = 0 \), and \( h(-1) = 1 \). Then \( h \in H^\infty \), and

\[
-1 \leq h(e^{i\theta}) < 0, \quad 0 \leq \theta < \pi/2
\]

\[
0 \leq h(e^{i\theta}) < 1, \quad \pi/2 \leq \theta < \pi
\]

\[
|h(e^{i\theta})| = 1, \quad \pi \leq \theta < 2\pi.
\]

Thus \( |k(e^{i\theta}) + h(e^{i\theta})| \leq 1 \), and \( h \) is the desired function.

Received by the editors December 8, 1966.

1 This research was supported in part by a National Science Foundation Grant. The author would like to thank Professor P. L. Duren, who suggested the problem in a course at the University of Michigan.

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POSITIVE $H^{1/2}$ FUNCTIONS ARE CONSTANTS

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The Koebe function $z/(1+z)^2$ is positive everywhere on $|z| = 1$, $z \neq -1$, and lies in the Hardy class $H^p$ for every $p < 1/2$. We show that this behavior is extreme by proving the following

**Theorem.** If $f(z) \in H^{1/2}$ and if $f(z) \geq 0$ a.e. on $|z| = 1$ then $f(z)$ is a constant.

**Proof.** We may assume that $f(z)$ is not identically 0. If $B(z)$ denotes the Blaschke product for the zeros of $f(z)$ then, as usual, we can write

$$f(z) = B(z)F^2(z), \quad F(z) \in H^1.$$  

We write the condition $f(z) \geq 0$ as the equation $f(z) = |f(z)|$ and conclude from (1) that

$$B(z)F^2(z) = |F^2(z)| \text{ a.e. on } |z| = 1.$$  

Since $f(z)$ is not identically 0 it follows that $F(z)$ is nonzero a.e. on $|z| = 1$. Thus we may divide (2) by $F(z)$ and obtain

$$B(z)F(z) = \overline{F(z)} \text{ a.e. on } |z| = 1.$$  

But the left side of (3) is $H^1$ and so has all negative Fourier coefficients 0, the right side is conjugate $H^1$ and so has all positive Fourier coefficients 0!.

Thus only the constant term remains and we conclude that both sides are constants. This is to say $B(z)F(z)$ and $F(z)$ are both constants and so indeed $f(z) = (B(z)F(z))$. $F(z)$ is a constant.

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Received by the editors May 17, 1967.