NONUNIQUENESS OF EXTREMAL KERNELS

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Given a "kernel" \( k = k(e^{i\theta}) \in L^q \), then

\[
\Phi(f) = \frac{1}{2\pi i} \int_{|z|=1} f(z) k(z) dz
\]

is a bounded linear functional on \( H^p \) \((1/p + 1/q = 1)\) with norm \( \|\Phi\| \leq \|k\|_q \). Two kernels induce the same linear functional, \( k_1 \sim k_2 \), if and only if they differ by an \( H^q \) function. It can be shown that the duality relation, \( \|\Phi\| = \inf \{\|k_i\|_q : k_1 \sim k_2\} \) holds for \( 1 \leq p \leq \infty \). One may ask whether an extremal kernel exists, i.e., a kernel \( k_1 \sim k \) such that \( \|\Phi\| = \|k_1\|_q \), and if so, whether it is unique.

By functional analytic methods, Havinson [1] and Rogosinski and Shapiro [2] showed that an extremal kernel always exists for \( 1 \leq p \leq \infty \) and is unique for \( 1 < p \leq \infty \). However, Rogosinski and Shapiro constructed a counterexample to show that it need not be unique for \( p = 1 \). As their example to show nonuniqueness for \( p = 1 \) was rather complicated, we present a simplification of their example.

Let

\[
k(e^{i\theta}) = \begin{cases} 
1, & 0 \leq \theta < \pi/2, \\
-1, & \pi/2 \leq \theta < \pi, \\
0, & \pi \leq \theta < 2\pi.
\end{cases}
\]

Rogosinski and Shapiro showed that this kernel induces a functional \( \Phi \) on \( H^1 \) with \( \|\Phi\| = 1 \), and thus \( k \) is an extremal kernel. To show this extremal kernel is not unique, it suffices to produce an \( H^\infty \) function \( h \neq 0 \) for which \( \|k + h\|_\infty = 1 \).

Let \( h \) be the conformal mapping sending the unit disk to the upper half-disk \( |w| < 1 \), \( \text{Im} \ w > 0 \), with \( h(1) = -1 \), \( h(i) = 0 \), and \( h(-1) = 1 \). Then \( h \in H^\infty \), and

\[
0 \leq h(e^{i\theta}) < 1, \quad \pi/2 \leq \theta < \pi
\]

\[
\left| h(e^{i\theta}) \right| = 1, \quad \pi \leq \theta < 2\pi.
\]

Thus \( \left| k(e^{i\theta}) + h(e^{i\theta}) \right| \leq 1 \), and \( h \) is the desired function.

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The Koebé function \( z/(1+z)^2 \) is positive everywhere on \( |z|=1 \), \( z \neq -1 \), and lies in the Hardy class \( H^p \) for every \( p < 1/2 \). We show that this behavior is extreme by proving the following

**Theorem.** If \( f(z) \in H^{1/2} \) and \( f(z) \geq 0 \) a.e. on \( |z|=1 \) then \( f(z) \) is a constant.

**Proof.** We may assume that \( f(z) \) is not identically 0. If \( B(z) \) denotes the Blaschke product for the zeros of \( f(z) \) then, as usual, we can write

\[
f(z) = B(z)F^2(z), \quad F(z) \in H^1.
\]

We write the condition \( f(z) \geq 0 \) as the equation \( f(z) = |f(z)| \) and conclude from (1) that

\[
B(z)F^2(z) = |F^2(z)| \quad \text{a.e. on } |z| = 1.
\]

Since \( f(z) \) is not identically 0 it follows that \( F(z) \) is nonzero a.e. on \( |z|=1 \). Thus we may divide (2) by \( F(z) \) and obtain

\[
B(z)F(z) = \overline{F(z)} \quad \text{a.e. on } |z| = 1.
\]

But the left side of (3) is \( H^1 \) and so has all negative Fourier coefficients 0, the right side is conjugate \( H^1 \) and so has all positive Fourier coefficients 0!.

Thus only the constant term remains and we conclude that both sides are constants. This is to say \( B(z)F(z) \) and \( F(z) \) are both constants and so indeed \( f(z) = (B(z)F(z)) \). \( F(z) \) is a constant.

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