The Koebe function $z/(1+z)^2$ is positive everywhere on $|z|=1$, $z \neq -1$, and lies in the Hardy class $H^p$ for every $p < 1/2$. We show that this behavior is extreme by proving the following

**Theorem.** If $f(z) \in H^{1/2}$ and $f(z) \geq 0$ a.e. on $|z|=1$ then $f(z)$ is a constant.

**Proof.** We may assume that $f(z)$ is not identically 0. If $B(z)$ denotes the Blaschke product for the zeros of $f(z)$ then, as usual, we can write

$$f(z) = B(z)F^2(z), \quad F(z) \in H^1.$$  

We write the condition $f(z) \geq 0$ as the equation $f(z) = |f(z)|$ and conclude from (1) that

$$B(z)F^2(z) = |F^2(z)| \quad \text{a.e. on } |z|=1.$$  

Since $f(z)$ is not identically 0 it follows that $F(z)$ is nonzero a.e. on $|z|=1$. Thus we may divide (2) by $F(z)$ and obtain

$$B(z)F(z) = \overline{F(z)} \quad \text{a.e. on } |z|=1.$$  

But the left side of (3) is $H^1$ and so has all negative Fourier coefficients 0, the right side is conjugate $H^1$ and so has all positive Fourier coefficients 0!.

Thus only the constant term remains and we conclude that both sides are constants. This is to say $B(z)F(z)$ and $F(z)$ are both constants and so indeed $f(z) = (B(z)F(z))$. $F(z)$ is a constant.