THE HAHN-BANACH EXTENSION AND THE LEAST UPPER BOUND PROPERTIES ARE EQUIVALENT

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In [1] it was proved that a finite dimensional partially ordered vector space $V$ has the Hahn-Banach extension property if and only if every nonempty set of elements in $V$ bounded from above has a least upper bound. The methods used in [1] are used to prove that the least upper bound property and the extension property are equivalent. The assumption of finite dimensionality is dropped.

The terminology and results of [1] are assumed. Some of these will be repeated for easy reference. Let $(V, C)$ be a partially ordered vector space (OLS) with positive wedge $C$, (i.e., $V$ is a real linear space with a nonempty subset $C$, such that $C+C \subseteq C$, $tC \subseteq C$, $t \geq 0$. The wedge $C$ determines an order relation, $u \preceq v$ if $u-v \in C$, which is transitive and $u \preceq v$ implies $tu \preceq tv$ and $u+w \preceq v+w$, $t \geq 0$, $w \in V$. The wedge $C$ is lineally closed if every line in $V$ intersects $C$ in a set which is closed relative to the line.

The OLS $(V, C)$ has the least upper bound property (LUBP) (or is a boundedly complete vector lattice) if every set of elements with an upper bound has a least upper bound (not necessarily unique). An OLS $(V, C)$ has the Hahn-Banach extension property (HBEP) if given (1) a real linear space $Y$, (2) a linear subspace $X$ of $Y$, (3) a function $p: Y \to V$ which is sublinear, (i.e., $p(y)+p(y') \geq p(y+y')$ and $p(ty) = tp(y)$, $y, y' \in Y$, $t \geq 0$) and (4) a linear function $f: X \to V$ such that $p(x) \geq f(x)$ for all $x \in X$, then there is a linear extension $F: Y \to V$ of $f$ such that $p(y) \geq F(y)$ for all $y \in Y$.

It is proved in [2], [3] that $(V, C)$ has the LUBP if and only if $(V, C)$ has the HBEP and $C$ is lineally closed. Thus the

**Theorem.** An OLS $(V, C)$ has the LUBP if and only if $(V, C)$ has the HBEP.

The theorem will be proved when it is shown that the HBEP for an OLS $(V, C)$ implies that $C$ is lineally closed. This was done in [1] under the assumption that $V$ was finite dimensional. In particular, in [1, Corollary 6.3] it is proved that a 2-dimensional OLS $(V, C)$ has the HBEP if and only if the wedge $C$ is lineally closed.

It is clear that a wedge $C$ in OLS $V$ is lineally closed if and only if the

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Intersection of $C$ with any 2-dimensional subspace $V_2$ of $V$ is lineally closed in $V_2$. Thus the theorem will be proved upon proof of the following lemma.

**Lemma.** If an OLS $(V, C)$ has the HBEP then every 2-dimensional ordered linear subspace $(V_2, C_2)$ of $(V, C)$ has the HBEP (where $C_2 = V_2 \cap C$ is the positive wedge in the linear subspace $V_2$ of $V$).

**Proof of Lemma.** Assume that $(V_2, C_2)$ is an ordered linear subspace of $(V, C)$ which does not have the HBEP. That is $C_2$ is not lineally closed. Then it can be assumed [1, p. 219] that $C_2$ is one of the following four sets:

$C^{(1)} = \{ v \in V_2, v = ab_1 + bb_2, \text{ where } a, b \in \mathbb{R}, \text{ and both } a > 0, b > 0 \text{ or both } a = 0, b = 0 \}$, (the open first “quadrant” of $V_2$ plus the origin);

$C^{(2)} = \{ v \in V_2, v = ab_1 + bb_2, \text{ where } a, b \in \mathbb{R}, \text{ and } a \neq 0, b > 0, \text{ or } a = 0, b = 0 \}$, (the first “quadrant” of $V_2$ excluding the open bounding ray through $b_1$);

$C^{(3)} = \{ v \in V_2, v = ab_1 + bb_2, \text{ where } b > 0, \text{ or } a = 0, b = 0 \}$, (the open upper half plane plus the origin);

$C^{(4)} = \{ v \in V_2, v = ab_1 + bb_2, \text{ where } b > 0, \text{ or } a \neq 0, b = 0 \}$, (the open upper half plane plus the closed bounding ray through $b_1$),

where $R$ is the real number field and $b_1, b_2$ determine an appropriate basis for $V_2$.

Note that the wedge $C_2^{(4)}$ is characterized by the property that $v \in C_2^{(4)}$ if and only if $-v \in C_2^{(4)}$ for every $v \neq 0, v$ in $V_2$.

**Case 1.** Let $C_2 = C_2^{(1)}$ or $C_2^{(2)}$ or $C_2^{(3)}$. We now refer to the specific example constructed in [1, p. 216, footnote (2)].

Let $Y = \mathbb{R}^3$, $X = \{(0, b, a) | b, a \in \mathbb{R}\}$. Define $f_2: X \to \mathbb{R}$ by $f_2(0, b, a) = a + b$ and $f_1: X \to \mathbb{R}$ by $f_1(0, b, a) = b$. Define $p_2: Y \to \mathbb{R}$ by

\[
p_2(t, b, a) = \begin{cases} |a| + b + t, & t \geq 0, b \geq 0 \smallskip \vspace*{1mm} \\ |a| + b + t + (a - t)^2/(a - t + b), & a > t, b \geq 0 \smallskip \vspace*{1mm} \\ |a| + t, & t \geq 0, b \geq 0 \smallskip \vspace*{1mm} \\ |a| + a, & a \geq 0, b \leq 0 \end{cases}
\]

Define:

\[
p: Y \to V_2, \quad p(z) = p_2(z)(b_1 + b_2), \quad z \in Y;
\]

\[
f: X \to V_2, \quad f(x) = f_1(x)b_1 + f_2(x)b_2, \quad x \in X.
\]

Observe $b_1 \in C_2$, so $b_1 \in C$.

Then by [1, p. 219, Case 1 and p. 220, Case 2(i)], $p$ is a sublinear function from $Y$ to $V_2$ with respect to the order determined by $C_2$. 

$f$ is linear from $X$ to $V_2$ and $p(x) - f(x) \in C_2$, $x \in X$. Further there is no linear extension $F$ of $f$ whose domain is all of $Y$, whose range is contained in $V_2$ and such that $p(y) - F(y) \in C_2$, $y \in Y$.

The assumption that $(V, C)$ has the HBEP guarantees that there is a linear extension $F$ of $f$ whose domain is $Y$ and whose range is a three dimensional subspace $V_3$ of $V$ which properly contains $V_2$ and such that $p(y) - F(y) \subset C_3$, $y \in Y$, where $C_3 = V_3 \cap C$. Consider $-F(1, 0, 0) = b_3$. Then $b_3 \neq 0$ and $\{b_1, b_2, b_3\}$ is a basis for $V_3$. Also,

(i) $p(t, b, a) - F(t, b, a) = (|a| + t)b_1 + (|a| - a + t)b_2 + tb_3 \subset C_3$, $t \geq a$, $b \geq 0$;

(ii) $p(t, b, a) - F(t, b, a) = (|a| + t + (a - t)^2/(a - t + b))b_1 + (|a| - a + t + (a - t)^2/(a - t + b))b_2 + tb_3 \subset C_3$, $a > t$, $b \geq 0$.

(a) Consider $a = 0$, $b = 0$ in (i). Then (i) implies that $t(b_1 + b_2 + b_3) \subset C_3$, $t \geq 0$.

(b) Consider $a = 0$, $b = t^2 + t$, $t \leq -1$ in (ii). Then (ii) implies that $t(b_1 + b_2 + b_3) \subset C_3$, $u \leq 0$.

Using the properties of a wedge, statements (a) and (b) imply that the line $L = \{t(b_1 + b_2 + b_3) \mid t \in \mathbb{R}\} \subset C_3 \subset \overline{C}$ (where $\overline{C}$ is the lineal closure of $C$) with the $\frac{1}{2}$-line for $t \geq 0$ in $C$ (by (a)) and the open $\frac{1}{2}$-line for $t < 0$ in $\overline{C}$ but possibly not in $C$: For letting $u$ approach $-\infty$ in (b), the resulting rays through $u(b_1 + b_2 + b_3) - b_3$ approach the ray through $-(b_1 + b_2 + b_3)$ and hence $-(b_1 + b_2 + b_3) \subset \overline{C}$.

If $L \subset C_3$, taking $a \geq 0$ in (i), it follows that $ab_1 + tb_1 + b_3 \subset C$ for $t \geq a$. Hence, by the wedge properties of $C_3$, $b_1 \subset C$, a contradiction.

Therefore, it must be assumed that $L^+ = \{t(b_1 + b_2 + b_3) \mid t \geq 0\} \subset C$ but $L^- = \{t(b_1 + b_2 + b_3) \mid t < 0\} \subset C$. Considering the subspace spanned by $L$ and $b_1 + b_2$, one obtains an induced wedge in this subspace of the type $C_2^{(4)}$. The next case considers the possibility of a wedge of this type.

Case 2. Assume $C_2$ has the form $C_2^{(4)}$. Then referring to [1, p. 221, Case (2v) and p. 217, Example 2] there is an example of a $C_2$-sublinear (hence $C$-sublinear) function $q: R_2 \to V_2$ and a linear function $f: X \to V_2$, where $X = \{(0, a) \mid a \in R\}$, which is $C_2$-dominated by $q$ and which has no linear extension $F$ with domain $R_2$ and range $V_2$ which is $C_2$-dominated by $q$. Specifically,

$$f(0, a) = ab_2, \quad a \in R;$$

$$q(y) = r_1(y)b_1 + r_2(y)b_2, \quad y \in R_2,$$

where

$$r_1(t, a) = -(at)^{1/2}, \quad t \geq 0 \text{ and } a \geq 0,$$

$$r_1(t, a) = 0, \quad t \leq 0 \text{ or } a \leq 0,$$
\begin{align*}
r_2(t, a) &= |a| + t, \quad t \geq 0, \\
r_2(t, a) &= a + at/(a - t), \quad t < 0 \text{ and } a > 0, \\
r_2(t, a) &= -a, \quad t \leq 0 \text{ and } a \leq 0.
\end{align*}

The assumption that \((V, C)\) has the HBEP implies that there exists a linear extension \(F\) of \(f\) with domain \(R_2\) and range in a linear subspace \(V_3\) of \(V\) with induced wedge \(C_3 = C \cap V_3\) where \(V_3 \neq V_2\) and \(F\) is \(C_3\)-dominated by \(q\). As in Case 1, set \(-F(1, 0) = b_3\). Then,

(i) \(q(t, a) - F(t, a) = -(at)^{1/2}b_1 + tb_2 + tb_3, \quad t \geq 0, \quad a \geq 0;\)

(iv) \(q(t, a) - F(t, a) = (at/(a - t))b_2 + tb_3, \quad t < 0, \quad a > 0.\)

(a) In (iv) set \(a = -dt, \quad d > 0\), then \((d/(d + 1))tb_2 + tb_3 \in C, \quad t < 0.\)

(b) In (i), set \(a = k^2/t\). Then \(-kb_1 + t(b_2 + b_3) \in C, \quad t > 0, \quad k \geq 0.\)

Let

\[Q_3 = \{ab_1 + \beta(b_2 + b_3) + \gamma b_2 | \quad \gamma > 0, \text{ or } \gamma = 0, \beta > 0, \text{ or } \gamma = 0, \beta = 0, \alpha \geq 0\};\]

Clearly, \(Q_3\) is a wedge. The wedge \(C_3 \supset Q_3\) for if \(\gamma > 0,\)

\[(1) \quad ab_1 + \gamma b_2 \in C_2 \subset C_3 \quad \text{for all } a,\]

and by (b),

\[(2) \quad -kb_1 + t(b_2 + b_3) \in C_2, \quad t > 0, \quad k \geq 0.\]

Adding (1) and (2),

\[(3) \quad \alpha b_1 + t(b_2 + b_3) + \gamma b_2 \in C_3\]

where \((a - k) = \alpha\) is arbitrary, and \(t > 0.\) By (a),

\[(4) \quad \beta b_2 + ((d + 1)/d)\beta b_3 \in C_3, \quad d > 0, \beta < 0.\]

Adding (1) and (4),

\[(5) \quad ab_1 + \beta(b_2 + b_3) + ((d + 1)/d)\beta b_3 \in C_3\]

where \(\gamma > 0, \beta < 0, \alpha\) is arbitrary, and \(d > 0.\) Consider any number \(\gamma' > 0, \quad d\) large enough and \(\gamma > 0\) so that \((d/(d + 1)) - 1)\beta + \gamma = \gamma'.\) Therefore,

\[(6) \quad ab_1 + \beta(b_2 + b_3) + \gamma' b_3 \in C_3, \quad \text{for } \gamma' > 0, \beta < 0, \alpha \text{ arbitrary.}\]

If \(\beta = 0,\) then

\[(7) \quad ab_1 + \gamma b_2 \in C_2 \subset C_3 \quad \text{for all } \gamma > 0, \text{ and all } \alpha.\]

Combining statements (3), (6) and (7),

\[(8) \quad \alpha b_1 + \beta(b_2 + b_3) + \gamma b_2 \in C, \quad \gamma > 0.\]
If $\gamma = 0$ and $\beta > 0$, then
\[(9) \quad ab_1 \in C_2 \subset C_3, \quad \text{for } a \geq 0.\]

By (b),
\[(10) \quad -kb_1 + \beta (b_2 + b_3) \in C_3, \quad \beta > 0, \quad k > 0.\]

Adding (9) and (10),
\[(11) \quad ab_1 + \beta (b_2 + b_3) \in C_3,\]

for arbitrary $\alpha = a - k, \beta > 0$.

If $\gamma = 0, \beta = 0$ and $ab_1 \in C_3$ then $ab_1 \in C_2$ and thus $\alpha \geq 0$. This statement plus (8) and (11) show that $Q_3 \subset C_3$.

It may be assumed that $Q_3 = C_3$. For otherwise it follows that $-b_1 \in C_3 \subset C$, a contradiction of the initial assumption in Case 2. To prove this, observe that the wedge $Q_3$ is characterized by the property that $v \in Q_3$ if and only if $-v \notin Q_3$, for every $v \in V_3, v \neq 0$. Equivalently, $Q_3$ consists of the open half-space of $V_3$ containing $b_2$ bounded by the subspace $V_2'$ spanned by $b_1$ and $b_2 + b_3$, joined with the open half-space in $V_2'$ containing $b_2 + b_3$ and bounded by the 1-dimensional subspace containing $b_1$, to which is adjoined the closed half-ray through $b_1$. If $v$ is an element in $C_3$ but not in $Q_3$, it then follows that $C_3$ either contains the closed half-space containing $Q_3$ (if $v \in V_2'$) or $C_3 = V_3$ (if $v \notin V_2'$). In both cases $-b_1 \in C_3$, the contradiction.

Let $(W, K)$ be an OLS where $K$ is a set such that every 2-dimensional linear subspace of $W$ intersects $K$ in a wedge of type $C_2^{(4)}$. The set $K$ is a wedge, for if $v_1$ and $v_2$ are in $K$ and $V_2'$ is a 2-dimensional linear subspace containing $v_1$ and $v_2$ then $V_2'$ cuts $K$ in a wedge (of type $C_2^{(4)}$) and so $v_1 + v_2$ and $\lambda v_1, \lambda \geq 0$, are in $V_2' \cap K \subset K$.

Further, $K$ must be a half-space. For if $v \in W$, let $V_2'$ be a 2-dimensional subspace of $W$ containing $v$. Then $V_2'$ cuts $K$ in a wedge of type $C_2^{(4)}$. Since such a wedge is characterized by the property that every nonzero vector (in its plane) or its negative, but not both, is in the wedge, it follows that $K$ has this property also. Since $K$ is convex, $K$ is a half-space.

The wedge $(V_3, C_3)$ (above) is an OLS of the same form as $(W, K)$. Additionally it is clear that the OLS $(W', K')$ where $W'$ is the subspace bounding $K$ and $K' = W' \cap K$ inherits the property that every 2-dimensional linear subspace of $W'$ intersects $K'$ in a wedge of type $C_2^{(4)}$.

The OLS $(W, K)$ does not have the HBEP. For if $B_2$ is an element of $K$ which is not in $W'$ and $B_1$ is an element of $K'$ which is not in the
hyperplane bounding \( K' \) in \( W' \), and \( r_1, r_2, f_1, f_2 \) are defined as in the beginning of Case 2, then

\[ q: R_2 \to W, \quad \text{where} \quad q(y) = r_2(y)B_2 + r_1(y)B_1, \quad y \in R_2, \text{is } K\text{-sublinear}; \]

\[ f: X \to W, \quad \text{where} \quad f(0, a) = aB_2, \quad a \in R, \text{ is linear and} \]

\[ q(x) - f(x) \in K, \quad x \in X. \]

But there is no linear extension \( F \) of \( f \) with domain \( R_2 \) and range \( W \) which is \( K \)-dominated by \( q \). If there were such an extension \( F \), the union of the ranges of \( q \) and of \( F \) would span a 3-dimensional subspace with basis \( B_2, B_1, B_3 \) where \( B_3 \) can be taken to be in the subspace \( W'' \) bounding \( K' \) in \( W' \). Note that if \( w = a_2B_2 + a_1B_1 + w'' \) with \( w'' \) in \( W'' \) and \( w \) in \( K \), then \( a_2 > 0 \) if \( a_2 = 0 \) then \( a_1 \geq 0 \).

Thus if \( F_i \) and \( q_i \) (\( i = 1, 2, 3 \)) are the coordinate functions of \( F \) and \( q \) respectively, then the fact that

\[ q(t, a) - F(t, a) = (q_2(t, a) - F_2(t, a))B_2 \]

\[ + (q_1(t, a) - F_1(t, a))B_1 - F_3(t, a)B_3 \]

is in \( K \) implies that \((q_2(t, a) - F_2(t, a))B_2 + (q_1(t, a) - F_1(t, a))B_1 \) is also in \( K \) for all \((t, a)\). Hence \( F' \), where \( F'(t, a) = F_2(t, a)B_2 + F_1(t, a)B_1 \), would be an extension of \( f \), a contradiction of the fact that \( f \) has no extension which is \( C_2 \)-dominated by \( q \) where \( C_2 \) is the induced wedge of type \( C^2(a) \) in the subspace \( V_2 \) spanned by \( B_1 \) and \( B_2 \).

If \((W, K)\) is an ordered linear subspace of \((V, C)\) then, since \((V, C)\) is assumed to have the HBEP, a linear extension \( F \) of \( f \) will exist with range contained in \( V \) which is \( C \)-dominated by \( q \). If \( W_3 \) is the subspace spanned by the union of the ranges of \( F \) and \( q \) with induced wedge \( K_3 = W_3 \cap K \), it follows that \((W_3, K_3)\) can be identified with \((V_3, C_3)\) considered previously (with \( C_3 = Q_3 \)) upon identifying \( B_1 \) with \( b_1 \), \( B_2 \) with \( b_2 \) and \( B_{1,5} \) with \( b_2 + b_3 \) so that

\[ K_3 = \{v \mid v = 0 \text{ or } v = a_1B_1 + a_{1,5}B_{1,5} + a_2B_2 \text{ where } a_2 > 0, \text{ or} \]

\[ a_2 = 0, a_{1,5} > 0, \text{ or } a_2 = 0, a_{1,5} = 0, a_1 > 0\}. \]

The ordered linear subspace \((\tilde{W}, \tilde{K})\) of \( V \) spanned by \( W \) and \( B_{1,5} \) has induced wedge \( \tilde{K} = \tilde{W} \cap C \) of the same type as \( K \). This will be proved when it is shown that if \( \tilde{w} \in \tilde{W} \) and \( \tilde{w} \neq 0 \), then \( \tilde{w} \in \tilde{K} \) if and only if \( -\tilde{w} \in \tilde{K} \). Let \( \tilde{w} = \alpha B_2 + \beta B_{1,5} + \gamma B_1 + w'' \), where \( w'' \in W'' \), the subspace bounding \( K' \), where \( K' \) is the wedge in \( W' \), the subspace bounding \( K \). If \( \alpha > 0 \), then \( \tilde{w} = u_1 + u_2 \), where \( u_1 = (\alpha/2)B_2 + \beta B_{1,5} \in K_3 \) and \( u_2 = (\alpha/2)B_2 + \gamma B_1 + w'' \in K \). Therefore \( \tilde{w} \in K_3 + K \subset \tilde{K} \). If \( \alpha < 0 \), and \( \tilde{w} \in K \), then from the previous sentence, \( u = -\alpha B_2 - \beta B_{1,5} \)
\(- (\gamma + 1)B_1 - w'' \in \tilde{K}\). Hence \(u + w = -B_1 \in \tilde{K} \cap W = K\), a contradiction, since \(B_1 \in K\).

If \(\alpha = 0, \beta > 0\), then

\[\tilde{w} = (\beta B_{1,5} + (\gamma - 1)B_1) + (B_1 + w'') \in K_8 + K \subset \tilde{K}.
\]

If \(\alpha = 0, \beta < 0\) and \(\tilde{w} \in \tilde{K}\), then from the previous sentence \(u = -\beta B_{1,5} - (\gamma + 1)B_1 - w'' \in \tilde{K}\). Therefore, \(u + w = -B_1 \in \tilde{K} \cap W = K\), again a contradiction.

If \(\alpha = 0, \beta = 0\), then \(\tilde{w} \in W\). Hence, if \(\tilde{w} \neq 0\), \(\tilde{w} \in K\) if and only if \(-\tilde{w} \in K\). Thus \(\tilde{K}\) has the form asserted.

A Zorn's Lemma argument guarantees the existence of a maximal ordered linear subspace \((W^*, K^*)\) of \((V, C)\) whose induced wedge \(K^*\) is of the same form as \(K\) and which does not have the HBEP. The previous argument proves that \((W^*, K^*) = (V, C)\), a contradiction. Hence, no 2-dimensional cut of \(C\) by a subspace is of the type \(C_2^{(4)}\).

Thus, every 2-dimensional cut of \(C\) is closed and the Lemma and the Theorem are proved.

**Bibliography**


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