A NOTE ON REFLEXIVE BANACH SPACES

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The purpose of this note is to prove three well-known theorems concerning reflexive Banach spaces by using exact sequences.

Let \( F \) denote either the field of real numbers or the field of complex numbers. Let \( B \) be the category whose objects are Banach spaces over the field \( F \) and whose morphisms are continuous linear maps \( T: X \to Y \). As usual, \( B(X, Y) \) denotes the set of all continuous linear maps from \( X \) to \( Y \). With the norm \( |T| = \sup_{|x| \leq 1} |T(x)| \) for each \( T \) in \( B(X, Y) \), \( B(X, Y) \) is a Banach space over \( F \). Notice that if \( T: X \to Y \) and \( S: Y \to Z \) then \( |ST| \leq |S| |T| \). This implies that

\[
B(T, Z): B(Y, Z) \to B(X, Z)
\]

is a morphism in the category \( B \) and \( |B(T, Z)| \leq |T| \). As in [1], we let \( B(X, F) = X^* \) and \( B(T, F) = T^* \).

Let \( Y \) be a closed subspace of the Banach space \( X \). Let \( i: Y \to X \) be the inclusion map. Then

\[
0 \to Y \xrightarrow{i} X \xrightarrow{p} Z \to 0,
\]

where \( Z = X/Y \) and \( p \) is the natural homomorphism, is an exact sequence in \( B \). By the Hahn-Banach Theorem, the sequence

\[
0 \to Y^* \xleftarrow{i^*} X^* \xleftarrow{p^*} Z^* \to 0
\]

is exact. Therefore the sequence

\[
0 \to Y^{**} \xrightarrow{i^{**}} X^{**} \xrightarrow{p^{**}} Z^{**} \to 0
\]

is also exact. Clearly, we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & Y \\
\downarrow & & \downarrow \\
0 & \to & Y^{**} \\
\downarrow n_1 & & \downarrow n_2 \\
0 & \to & X \xrightarrow{i} X^{**} \xrightarrow{p^{**}} Z^{**} \to 0
\end{array}
\]

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where \( n_1, n_2, n_3 \) are the natural embeddings and all the rows and columns are exact. Therefore, by “diagram chasing,” we have the following commutative diagram (D):

\[
\begin{array}{ccc}
0 & \to & Y \\
\downarrow & & \downarrow i \\
0 & \to & X & \to & Z & \to & 0 \\
\downarrow n_1 & & \downarrow i^* & & \downarrow p \\
0 & \to & Y^{**} & \to & X^{**} & \to & Z^{**} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Y^{**}/Y & \to & X^{**}/X & \to & Z^{**}/Z & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

where all the rows and columns are exact. In particular, the following sequence (E) is exact.

\[(E) \quad 0 \to Y^{**}/Y \to X^{**}/X \to Z^{**}/Z \to 0.\]

We observe that the Banach space \( X \) is reflexive if and only if the Banach space \( X^{**}/X \) in (D) (or (E)) is equal to 0.

**Theorem 1.** If \( X \) is a reflexive Banach space and \( Y \) is a closed subspace of \( X \), then \( Y \) is reflexive.

**Proof.** By the exactness of the sequence (E), we have \( X \) is reflexive \( \Rightarrow X^{**}/X = 0 \Rightarrow Y^{**}/Y = 0 \Rightarrow Y \) is reflexive.

**Theorem 2.** If \( X \) is a Banach space and \( Y \) is a closed subspace of \( X \), and if both \( Y \) and \( X/Y \) are reflexive, then \( X \) is reflexive.

**Proof.** Again by the exactness of the sequence (E), we have \( Y \) and \( X/Y \) are reflexive \( \Rightarrow Y^{**}/Y = 0 \) and \( Z^{**}/Z = 0 \Rightarrow X^{**}/X = 0 \Rightarrow X \) is reflexive.

**Theorem 3.** If \( X \) is a reflexive Banach space and \( Y \) is a closed subspace of \( X \), then \( X/Y \) is reflexive.

**Proof.** By the exactness of the sequence (E), we have \( X \) is reflexive \( \Rightarrow X^{**}/X = 0 \Rightarrow Z^{**}/Z = 0 \Rightarrow Z = X/Y \) is reflexive.
ON TYPE I C*-ALGEBRAS

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1. Introduction. Recently, the author [4] proved the equivalence of type I C*-algebras and GCR C*-algebras without the assumption of separability. On the other hand, for separable type I C*-algebras, we have a simpler criterion as follows: a separable C*-algebra $\mathfrak{A}$ is of type I if and only if every irreducible image contains a nonzero compact operator.

It has been open whether or not this remains true when $\mathfrak{A}$ is not separable (cf. [1], [2], [3]).

In the present paper, we shall show that a C*-algebra $\mathfrak{A}$ is GCR if and only if every irreducible image contains a nonzero compact operator, so that by the author’s previous theorem [4], the above problem is affirmative for arbitrary C*-algebra.

2. Theorem. In this section, we shall show the following theorem.

Theorem. A C*-algebra $\mathfrak{A}$ is of type I if and only if every irreducible image contains a nonzero compact operator.

Proof. Suppose that a C*-algebra $\mathfrak{A}$ is of type I, then it is GCR and so every irreducible image contains a nonzero compact operator (cf. [1], [2], [3], [4]).

Conversely suppose that every irreducible image of $\mathfrak{A}$ contains a nonzero compact operator. It is enough to assume that $\mathfrak{A}$ has the unit $I$. We shall assume that $\mathfrak{A}$ is not of type I. Then it is not GCR; then there is a separable non-type I C*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (cf. [2], [4]). Take a pure state $\phi$ on $\mathfrak{B}$ such that the image of $\mathfrak{B}$ under the irreducible *-representation $\{U_\phi, \mathfrak{H}_\phi\}$ of $\mathfrak{B}$ constructed via $\phi$ does not contain any nonzero compact operator, where $\mathfrak{H}_\phi$ is a Hilbert space.

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