A NOTE ON REFLEXIVE BANACH SPACES

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The purpose of this note is to prove three well-known theorems concerning reflexive Banach spaces by using exact sequences.

Let F denote either the field of real numbers or the field of complex numbers. Let $\boldsymbol{B}$ be the category whose objects are Banach spaces over the field F and whose morphisms are continuous linear maps $T: X \to Y$. As usual, $\boldsymbol{B}(X, Y)$ denotes the set of all continuous linear maps from X to Y. With the norm $|T| = \sup_{|x| \leq 1} |T(x)|$ for each T in $\boldsymbol{B}(X, Y)$, $\boldsymbol{B}(X, Y)$ is a Banach space over F. Notice that if $T: X \to Y$ and $S: Y \to Z$ then $|ST| \leqq |S|\,|T|$. This implies that

$$\boldsymbol{B}(T, Z): \boldsymbol{B}(Y, Z) \to \boldsymbol{B}(X, Z)$$

is a morphism in the category $\boldsymbol{B}$ and $|\boldsymbol{B}(T, Z)| \leqq |T|$. As in [1], we let $\boldsymbol{B}(X, F) = X^*$ and $\boldsymbol{B}(T, F) = T^*$.

Let Y be a closed subspace of the Banach space X. Let $i: Y \to X$ be the inclusion map. Then

$$0 \to Y \xrightarrow{i} X \xrightarrow{p} Z \to 0,$$

where $Z = X/Y$ and p is the natural homomorphism, is an exact sequence in $\boldsymbol{B}$. By the Hahn-Banach Theorem, the sequence

$$0 \leftarrow Y^* \xleftarrow{i^*} X^* \xleftarrow{p^*} Z^* \leftarrow 0$$

is exact. Therefore the sequence

$$0 \longrightarrow Y^{**} \xrightarrow{i^{**}} X^{**} \xrightarrow{p^{**}} Z^{**} \longrightarrow 0$$

is also exact. Clearly, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, "n_1"'] & X \arrow[r, "p"] \arrow[d, "n_2"'] & Z \arrow[r] \arrow[d, "n_3"'] & 0 \\
0 \arrow[r] & Y^{**} \arrow[r, "i^{**}"] & X^{**} \arrow[r, "p^{**}"] & Z^{**} \arrow[r] & 0
\end{tikzcd}$$

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where n_1, n_2, n_3 are the natural embeddings and all the rows and columns are exact. Therefore, by "diagram chasing," we have the following commutative diagram (D):

$$
\text{(D)}\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, "n_1"'] & X \arrow[r, "p"] \arrow[d, "n_2"'] & Z \arrow[r] \arrow[d, "n_3"'] & 0 \\
0 \arrow[r] & Y^{**} \arrow[r, "i^{**}"] \arrow[d] & X^{**} \arrow[r, "p^{**}"] \arrow[d] & Z^{**} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y^{**}/Y \arrow[r] \arrow[d] & X^{**}/X \arrow[r] \arrow[d] & Z^{**}/Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

where all the rows and columns are exact. In particular, the following sequence (E) is exact.

$$
\text{(E)}\qquad 0 \to Y^{**}/Y \to X^{**}/X \to Z^{**}/Z \to 0.
$$

We observe that the Banach space X is reflexive if and only if the Banach space X^{**}/X in (D) (or (E)) is equal to 0.

THEOREM 1. *If X is a reflexive Banach space and Y is a closed subspace of X, then Y is reflexive.*

PROOF. By the exactness of the sequence (E), we have X is reflexive $\Rightarrow X^{**}/X=0 \Rightarrow Y^{**}/Y=0 \Rightarrow Y$ is reflexive.

THEOREM 2. *If X is a Banach space and Y is a closed subspace of X, and if both Y and X/Y are reflexive, then X is reflexive.*

PROOF. Again by the exactness of the sequence (E), we have Y and X/Y are reflexive $\Rightarrow Y^{**}/Y=0$ and $Z^{**}/Z=0 \Rightarrow X^{**}/X=0 \Rightarrow X$ is reflexive.

THEOREM 3. *If X is a reflexive Banach space and Y is a closed subspace of X, then X/Y is reflexive.*

PROOF. By the exactness of the sequence (E), we have X is reflexive $\Rightarrow X^{**}/X=0 \Rightarrow Z^{**}/Z=0 \Rightarrow Z=X/Y$ is reflexive.

REFERENCE

1. N. Dunford and J. T. Schwartz, *Linear operators*, Vol. I, Interscience, New York, 1958.

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ON TYPE I C^*-ALGEBRAS

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1. **Introduction.** Recently, the author [4] proved the equivalence of type I C^*-algebras and GCR C^*-algebras without the assumption of separability. On the other hand, for separable type I C^*-algebras, we have a simpler criterion as follows: a separable C^*-algebra $\mathfrak{A}$ is of type I if and only if every irreducible image contains a nonzero compact operator.

It has been open whether or not this remains true when $\mathfrak{A}$ is not separable (cf. [1], [2], [3]).

In the present paper, we shall show that a C^*-algebra $\mathfrak{A}$ is GCR if and only if every irreducible image contains a nonzero compact operator, so that by the author's previous theorem [4], the above problem is affirmative for arbitrary C^*-algebra.

2. **Theorem.** In this section, we shall show the following theorem.

THEOREM. *A C^*-algebra $\mathfrak{A}$ is of type* I *if and only if every irreducible image contains a nonzero compact operator.*

PROOF. Suppose that a C^*-algebra $\mathfrak{A}$ is of type I, then it is GCR and so every irreducible image contains a nonzero compact operator (cf. [1], [2], [3], [4]).

Conversely suppose that every irreducible image of $\mathfrak{A}$ contains a nonzero compact operator. It is enough to assume that $\mathfrak{A}$ has the unit I. We shall assume that $\mathfrak{A}$ is not of type I. Then it is not GCR; then there is a separable nontype I C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (cf. [2], [4]). Take a pure state ϕ on $\mathfrak{B}$ such that the image of $\mathfrak{B}$ under the irreducible *-representation $\{U_\phi, \mathfrak{H}_\phi\}$ of $\mathfrak{B}$ constructed via ϕ does not contain any nonzero compact operator, where $\mathfrak{H}_\phi$ is a Hilbert space.

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