A REMARK ON TRANSITIVITY IN BUNDLES

D. C. WEST

A regular (weakly) transitive translation function for a fibre bundle induces an H-space antihomomorphism from the loop space of the base into the group of the bundle. The question when such translation functions exist has had three answers to date. E. Brown [**1**, p. 226] states that every bundle with paracompact base admits weakly transitive translation functions. However, E. Fadell has pointed out an essential error in the proof. (Namely, [**1**, p. 227] $\nu_\omega(\alpha)$, $\nu_\omega(\beta)$ and $\nu_\omega(\alpha+\beta)$ are unrelated.) In Steenrod's book [**4**, p. 59] we find that if a bundle has totally disconnected group, then it admits transitive translation functions. J. Schlesinger [**3**] has proved a converse to this result for a restricted class of bundles. Schlesinger's theorem fails to contradict Brown's claim for the following reasons: (1) Brown employs Moore paths while Schlesinger uses ordinary (unit domain) paths. (2) Schlesinger discusses transitivity while Brown claims only weak transitivity.

This paper removes the first of these distinctions; namely we prove:

THEOREM. *A fibre space admits a regular transitive translation function if and only if it admits a regular transitive Moore translation function.*

Definitions and notation. The word map will mean continuous transformation and all spaces will be assumed to be Hausdorff.

In discussing the path space B^I, we will use the notation of Schlesinger and denote translation functions by τ; paths will be $\omega\in B^I$ and path composition will be written $\omega_1 \circ \omega_2$. Given a path $\omega\in B^I$ and translation function τ, τ_ω will be defined by $\tau_\omega(e)=\tau(e, \omega)$.

Given $r\geqq 0$ and $a\geqq 0$, the map r^*: $[0, a]\to[0, ra]$ will be defined by $r^*(t)=rt$.

The Moore path space is the set $P(B)=\{\alpha|\alpha\colon [0, r]\to B,\ r\geqq 0\}$ topologized by letting h: $P(B)\to B^I\times R$ (where $h(\alpha)=(\alpha\cdot r^*, r)$) be an embedding. Elements of $P(B)$ will be α: $[0, r]\to B$ and β: $[0, s]\to B$ and the path operation will be written $\alpha+\beta$ where $\alpha+\beta$: $[0, r+s]\to B$. Note that when we suppress the domain of $\alpha\in P(B)$ (see Brown [**1**, p. 224]) we have $B^I\subset P(B)$. A Moore translation function for p: $E\to B$ is a map τ': $\{(e, \alpha)\in E\times P(B)\,|\,p(e)=\alpha(0)\}\to E$ such that $p\tau'(e, \alpha)=\alpha(r)$. We will denote all Moore translation functions by τ'.

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Such a τ' is regular if $\tau'(e, \alpha)=e$ whenever α is constant and it is subregular if $\tau'(e, \alpha_0)=e$ for α_0: $[0, 0]\to p(e)$.

A Moore translation function is said to be transitive provided $\tau'_{\alpha+\beta}=\tau'_\beta \cdot \tau'_\alpha$.

Main results.

THEOREM 1. *A space p: $E\to B$ over B admits a regular, transitive translation function τ if and only if it allows a regular, transitive Moore translation function τ'.*

PROOF. Suppose τ is regular and transitive for B^I. Define $\tau'(e, \alpha) =\tau(e, a\cdot r^*)$; clearly this is a regular Moore translation function. To prove that τ' is transitive, we consider two paths α and β in $P(B)$ and their composition $\alpha+\beta$. If either α or β is a trivial path, the transitivity follows from the regularity of τ. In the more general case (when $rs\neq 0$), we notice that $(\alpha+\beta)\cdot(r+s)^*$ is a reparameterization of $(\alpha\cdot r^*)\circ(\beta\cdot s^*)$, so the transitivity of τ' is implied by the following lemma of Schlesinger [**3**, p. 508].

LEMMA 2. *If τ is a transitive translation function for p: $E\to B$ and f: $[0, 1]\to[0, 1]$ is a sense-preserving homeomorphism (reparameterization) then $\tau_{\omega f}=\tau_\omega$.*

This completes the first half of the theorem.

To prove the converse, suppose that τ' is a regular transitive Moore translation function and define $\tau(e, \omega)=\tau'(e, \omega)$. Again we have a regular translation function and it remains to show that τ is transitive.

We see that $\tau(e, \omega_1\circ\omega_2)=\tau'(e, \omega_1\circ\omega_2)$ and $\tau(\tau(e, \omega_1), \omega_2) =\tau'(\tau'(e, \omega_1), \omega_2)$. Since $\omega_1\circ\omega_2=(\omega_1+\omega_2)\cdot 2^*$ the proof will be completed by the following lemma.

LEMMA 3. *If τ' is a regular transitive Moore translation function and α: $[0, r]\to B$, then $\tau'(e, \alpha\cdot 2^*)=\tau'(e, \alpha)$.*

PROOF. Given $\epsilon>0$, it suffices to construct f: $[0, r]\to[0, r]$ which is an ϵ approximation to the identity map and such that $\tau'(e, \alpha\cdot f) =\tau'(e, \alpha\cdot 2^*)$. We will then conclude $\tau'(e, \alpha\cdot 2^*)=\tau'(e, \alpha)$ by the continuity of τ'. To that end let $r/n<\epsilon$ and define

$$\begin{aligned} f(r(i-1+t)/n) &= r(i-1+2t)/n && \text{if } 0\leqq t\leqq 1/2,\\ &= ri/n && \text{if } \tfrac{1}{2}\leqq t\leqq 1 \end{aligned}$$

for $i=1, 2, \cdots, n$.

Now express α as the sum of n paths α_i: $[0, r/n]\to B$ and note that $\alpha\cdot f$ will be expressible as $\alpha\cdot f=(\alpha_1\cdot 2^*)+\beta_1+(\alpha_2\cdot 2^*)+\beta_2+\cdots$

$+(\alpha_n \cdot 2^*)+\beta_n$ where the β_i are suitable constant paths. Also $\alpha \cdot 2^* = (\alpha_1 \cdot 2^*)+(\alpha_2 \cdot 2^*)+ \cdots +(\alpha_n \cdot 2^*)$ so by the regularity and transitivity of τ' we see that $\tau'(e, \alpha \cdot f) = \tau'(e, \alpha \cdot 2^*)$, so the lemma and theorem are now finished.

We observe that in the previous constructions, the properties that τ_ω (or τ_α') is a homeomorphism and that $\Phi_i^{-1}(\omega(1))\tau_\omega \Phi_j(\omega(0))$ (or $\Phi_i^{-1}(\alpha(r))\tau_\alpha' \Phi_j(\alpha(0))$) is in the group of a fibre bundle, are preserved. Hence we may restate Schlesinger's theorem for Moore paths as follows:

THEOREM 4. *A fibre bundle over a finite polyhedron with structural group G, which has no small subgroups, has a regular transitive Moore translation function if and only if it is equivalent in G to an H bundle where H is a totally disconnected subgroup of G.*

REMARK. Thus there are numerous examples of bundles which do not admit regular transitive Moore translation functions.

Comment on Lemma 2. Schlesinger's lemma (Lemma 2 above) is interesting in its own right and as we shall see, its restatement for lifting functions is also true. However, if we translate to Moore paths, the pattern changes somewhat. For instance, any transitive lifting function for B^I is necessarily regular. Curiously, the corresponding statement for Moore paths is false, as the following lemma shows.

LEMMA 5. *Suppose λ' is a transitive Moore lifting function for p: $E \to B$. Then λ' is regular if and only if it commutes with reparameterizations (i.e., iff for any sense-preserving homeomorphism f: $[0, s] \to [0, r]$, $\lambda'(e, \alpha \cdot f) = \lambda'(e, \alpha) \cdot f$).*

This may be proved by approximating f by a piecewise linear map where the maps on the pieces reduce to $(2^k)^*$. Then we need only prove the special case $\lambda'(e, \alpha \cdot 2^*) = \lambda'(e, \alpha) \cdot 2^*$. The proof of this is analogous to the proof of the similar lemma for translation functions.

The same methods, combined with the fact that transitive lifting functions for B^I are regular, give us the following restatement of Lemma 2.

LEMMA 6. *A transitive lifting function for B^I commutes with reparameterizations.*

Also, if τ' is a transitive translation function for $P(B)$, we have a retraction R: $E \to E_0$ given by $R(e) = \tau'(e, \alpha_0)$, and τ' may be factored into R and the restriction τ_0' of τ' to E_0; $\tau'(e, \beta) = \tau'(e, \alpha_0 + \beta) = \tau'(\tau'(e, \alpha_0), \beta) = \tau_0'(R(e), \beta)$. If τ_0' is invariant under reparameteri-

zations, then so is τ' (and conversely) and τ_0' is transitive iff τ' is transitive. Notice that τ_0' is subregular and that the conversions $\tau' \to \lambda' \to \tau''$ given by $\lambda'(e, \alpha)(t) = \tau'(e, \alpha|_{[0,t]})$ and $\tau''(e, \alpha) = \lambda'(e, \alpha)(1)$ preserve regularity, transitivity, and invariance under reparameterizations, and that the two conversions together give $\tau'' = \tau'$. Finally, any subregular translation function which commutes with reparameterizations is regular (consider α as r approaches zero). These considerations combine to give the following variant of Lemma 5.

LEMMA 7. *A transitive translation function τ' for $P(B)$ is invariant under reparameterizations iff its subregular restriction τ_0' (defined above) is regular.*

There are many examples of regular transitive lifting functions, but to show that Lemma 5 is nonvacuous, we should exhibit a transitive, nonregular Moore lifting function.

Let B be the union of uncountably many intervals with a common endpoint, such that a neighborhood of that endpoint contains all but finitely many of the intervals. The fibre space is the classical fibration $p: P(B) \to B$ where $p(\alpha) = \alpha(r)$. A transitive lifting function may be defined as follows. If β: $[0, s] \to B$ and $\alpha(r) = \beta(0)$, let $\lambda'(\alpha, \beta)(t) = \alpha + (\beta|_{[0,t]})$. Then λ' is transitive because the Moore path operation is associative. If any Moore lifting function λ' were regular for this fibre space, we would be able to produce a regular lifting function for the corresponding space p: $B^I \to B$. However, this fibre space, constructed by Tully [5], is an example of a fibration allowing no regular lifting function. (It is in fact *the* example since Tully has shown in [5] that the usual example due to Hurewicz [2] actually allows regular lifting functions.)

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