1. Introduction. Right alternative rings arise when the alternative identity is weakened [1]. That is, a ring $R$ is called right alternative if the identity $(y, x, x)=0$ is satisfied for all $x, y$ in $R$ where the associator is defined as $(x, y, z) = (xy)z-x(yz)$. When the characteristic of $R$ is prime to 2 this is equivalent to the identity

$$ (x, y, z) + (x, z, y) = 0 $$

for all $x, y, z \in R$.

Many authors have investigated right alternative rings (see the bibliography). In this paper we examine a subclass of these rings, the $(-1, 1)$ rings. Such rings $R$ satisfy (1) and the identity

$$ (x, y, z) + (y, z, x) + (z, x, y) = 0 $$

for all $x, y, z \in R$.

Maneri [7] proved that a simple ring of type $(-1, 1)$ with characteristic prime to 6 having an idempotent $e \neq 0, 1$ is associative. It is shown in this paper that when $R$ is a $(-1, 1)$ ring with no trivial ideals which has characteristic prime to 6, then if $R$ contains an idempotent $e \neq 0, 1$, it has a Peirce decomposition relative to $e$. Further, the multiplicative relations between the submodules of the Peirce decomposition relative to containment are the same as those for an associative ring. Under the additional assumption that $R$ is a prime ring it is proven that $R$ must be associative.

2. Preliminary section. A Peirce decomposition. We will assume throughout this section that $R$ is a $(-1, 1)$ ring with characteristic prime to 6 having an idempotent $e \neq 0, 1$. When other conditions on $R$ are needed they will be noted.

The commutator is defined as $(x, y) = xy - yx$ where $x, y \in R$. It is simple to verify that the identity $C(x, y, z) = (xy, z) - x(y, z) - (x, z)y - (x, y, z) + (z, x, y) = 0$ is satisfied by all elements $x, y, z$ in an arbitrary ring. When $R$ satisfies (1) this reduces to

$$ C(x, y, z) = (xy, z) - x(y, z) - (x, z)y - 2(x, y, z) - (z, x, y) = 0. $$

The following identities hold in an arbitrary right alternative ring [5].

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(4) \( J(x, w, y, z) = (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w = 0 \),
(5) \( K(x, y, z) = (x, y^2, z) - (x, y, yz + zy) = 0 \).

When \( R \), in addition, satisfies (2) and has an idempotent \( e \), the following identities are satisfied for all \( x, y, w \in R \) [7].

(6) \( (x, (e, e, y)) = 0 \),
(7) \( (e, e, y)(e, e, w) = 0 \),
(8) \( (e, e, (e, x))y = (e, e, y(e, x)) \).

Next, define \( U = \{ u \in R \mid (u, R) = 0 \} \). Then if \( u \) is in \( U \), \( 0 = C(x, x, u) = -2(x, x, u) \) and so

(9) \( (x, x, u) = 0 \).

From (1) it then follows that \( (x, u, x) = 0 \). If \( x \) is replaced by \( x+y \) in (9) and the last equation we obtain

(10) \( (x, y, u) = - (y, x, u), \quad (x, u, y) = - (y, u, x) \).

We will find the Teichmüller identity, which follows, useful. For all \( x, y, z, w \in R \), where \( R \) is an arbitrary ring,

\[
F(x, y, z, w) = (xy, z, w) - (x, yz, w) + (x, y, zw) - x(y, z, w) - (x, y, z)w = 0.
\]

**Lemma 1.** Let \( A \neq 0 \) be an ideal of \( R \). Then the set of two-sided annihilators of \( A \) is an ideal of \( R \).

**Proof.** Suppose \( x \) is in \( R \) and \( xA = Ax = 0 \). Let \( a \) belong to \( A \) and \( y \) to \( R \). From (1) we get \( 0 = (y, a, x) + (y, x, a) = (y, x, a) = (yx)a \) and \( 0 = (a, y, x) + (x, y, a) = (x, y, a) = (xy)a \). Then from (2), \( 0 = (a, x, y) + (x, y, a) + (y, a, x) = (a, x, y) + (x, y, a) = -a(xy) + (xy)a = -a(xy) \) and \( 0 = (a, y, x) + (y, x, a) + (x, a, y) = (a, y, x) + (y, x, a) = -a(yx) + (yx)a = -a(yx) \). Hence, the two-sided annihilators of \( A \) form an ideal of \( R \).

The next lemma is crucial to the existence of a Peirce decomposition relative to \( e \).

**Lemma 2.** Let \( R \) have characteristic prime to 6. Then let \( s \) belong to \( R \) such that \( s \) and \( sR \) belong to \( U \). Define

\[
B_s = \{ x \in R \mid xs = x(sR) = (xR)s = (Rx)s = (xR)(Rs) = (Rx)(Rs) = 0 \}.
\]

Then \( B_s \) is an ideal of \( R \).
Proof. Let $x \in B$, and $y, z, w \in R$. From the definition of $B$, it is immediate that

\[(12) \quad (xw)s = (wx)s = (xw)(ys) = (wx)(ys) = 0.\]

Now $(wx, y, s) = -(y, wx, s)$ and $(xw, y, s) = -(y, xw, s)$ from (10) since $s$ is in $U$. Expanding these associators, we have $[(wx)y]_s = -[y(wx)]_s$ and $[(xw)y]_s = -[y(xw)]_s$. However, $(wx, y, s) = -(wx, s, y) = 0$ and $(xw, y, s) = -(xw, s, y) = 0$ from (12). We conclude that

\[(13) \quad [(wx)y]_s = [(xw)y]_s = [y(wx)]_s = [y(xw)]_s = 0.\]

It remains to show that $[(xw)y]_s = [(xw)y]_s = [y(xw)]_s = [y(xw)]_s = 0$. We will show that the expressions involving $xw$ vanish, and note that an identical proof applies to the remaining two expressions. We have, from (4), $0 = J(xw, s, y, z) = (xw, s, yz) + (xw, y, sz) - (xw, s, z)y - (xw, y, z)s$. Since $(xw, s, yz) = (xw, s, z)y = 0$ by (12) and (13) we obtain $(xw, y, sz) = (xw, y, z)s$. Combining this equation with the fact that $sz$ is in $U$ we get $(xw, y, sz) = -(xw, sz, y) = (y, sz, xw) = (xw, y, z)s$. Expanding the latter three associators and using (12) and (13) we obtain

\[(14) \quad [y(sz)](xw) = (xw)[(sz)y] = \{[(xw)y]z\}s.\]

Next, we have $0 = (xw, y, sz) + (y, sz, xw) + (sz, xw, y) = -2(xw, sz, y) - (sz, y, xw) = 2(xw)[(sz)y] - [(sz)y](xw) + (sz)[y(xw)]$ from (2), (10) and (12). Then, from (14) and the fact that $sz \in U$, we obtain

\[(15) \quad (xw)[(sz)y] = -(sz)[y(xw)].\]

Now, from (1), $(xw, y, sz) = -(xw, sz, y)$. Expanding these associators and using (12) we obtain $[(xw)y](sz) - (xw)[y(sz)] = (xw)[(sz)y]$. Since $sz$ belongs to $U$ this equation becomes

\[(16) \quad 2(xw)[(sz)y] = [(xw)y](sz).\]

Combining (15) and (16) we get

\[(17) \quad 2(sz)[y(xw)] + [(xw)y](sz) = 0.\]

From (14) and (16) and the fact that $s$ belongs to $U$ we have

\[(18) \quad 2\{[(xw)y]z\}s = [(xw)y](zs).\]

It then follows from (18) that $((xw)y, z, s) = -(xw)y, s, z) = [(xw)y](sz)$ from (13) and so, $[(xw)y](sz) = -[(xw)y]z$ from (18). This equation, combined with (18), yields $[(xw)y](zs) = 0$ when divided by 3. Whence, from (17) and the
The fact that $sz$ is in $U$, it follows that $[y(xw)](sz) = 0$. Thus $B_s$ is an ideal of $R$.

We now assume that $R$ has no trivial ideals. The following lemma leads directly to the existence of the Peirce decomposition.

**Lemma 3.** $(e, e, (e, x)) = 0$ for all $x$ in $R$.

**Proof.** Let $(e, e, (e, x)) = b$. Then from (6) we obtain $(b, R) = 0$. Furthermore, $(bR, R) = 0$ from (8). Thus the element $b$ satisfies the requirements for the element $s$ of Lemma 2.

On the other hand, it is quite clear from (6), (7), and (8) that $b$ also belongs to $B_b$. Now let $C$ be the ideal generated by $b$. Then $C$ is contained in $B_b$. But from Lemma 2 it is evident that $B_bB_b = B_b = 0$. Then from Lemma 1 it follows that since the two-sided annihilators of an ideal form an ideal of $R$, $CB_b = B_bC = 0$ and so $C^2 = 0$. That is, $C$ is a trivial ideal of $R$. Hence, $C = 0$.

**Theorem 1.** Let $R$ be a $(-1, 1)$ ring with no trivial ideals. Further, suppose that the characteristic of $R$ is prime to 6. Then if $R$ has an idempotent, $e$, $R$ has the desired Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ where $x$ belongs to $R_{ij}$ if and only if $ex = ix$ and $xe = jx$ for $i, j = 0, 1$ and the sum of the submodules is direct.

**Proof.** It suffices to show that $(e, e, x) = (e, x, e) = (x, e, e) = 0$ for all $x$ in $R$. From the fact that $R$ is right alternative, $(x, e, e) = 0$ for all $x$ in $R$. Next, from (5), $0 = K(e, e, x) = (e, e, x) - (e, e, ex + xe)$. Since $(e, e, (e, x)) = 0$ from Lemma 3 we obtain $(e, e, x) - 2(e, e, ex) = 0$. Replacing $x$ with $ex$ in the last equation we get $(e, e, ex) - 2(e, e, e(ex)) = 0$. Since $(e, e, x)$ is in $U$, it follows from (9) with $e$ and $(e, e, x)$ substituted for $x$ and $u$ that $0 = (e, e, (e, e, x)) = (e, e, ex) - (e, e, e(ex))$. Thus, $(e, e, ex) = 0$. But then $(e, e, x) = 0$. Finally, from (1), $(e, x, e) = 0$ and $R$ has the desired Peirce decomposition relative to $e$.

3. **Main section.** Let $R$ be a $(-1, 1)$ ring with no trivial ideals containing an idempotent $e \neq 0, 1$. Under the assumption that the characteristic of $R$ is prime to 6 the following two lemmas are satisfied by $R [7]$.

**Lemma 4.** The following multiplication table, with respect to containment, holds for the submodules $R_{ij}$ of the Peirce decomposition of $R$.

$$R_{ij}R_{km} \subseteq \delta_{jk}R_{im} \text{ except } R_{ij} \subseteq R_{ii} \text{ where } i, j, k, m = 0, 1 \text{ and }$$

$$\delta_{jk} \text{ is the Kronecker delta.}$$

**Lemma 5.** For all $x, y$ in $R$ and the idempotent $e$ the following identity holds.
We now proceed to examine the submodules $R_{10}$ and $R_{01}$ carefully. We will show that $R_{ij} = 0$ when $i = 0$, 1 and $j = 1 - i$ and that $R_{ij}$ is in the center of $R$. It is then easy to show that $R_{ij}$ is a trivial ideal of $R$ and hence must be zero.

**Lemma 6.** $R_{ij} = 0$ for $i = 0$, 1 and $j = 1 - i$.

**Proof.** Let $x_{ij}$, $y_{ij}$, $z_{ij}$ belong to $R_{ij}$. Then

$$0 = (y_{ij}, x_{ij}, x_{ij}) = (y_{ij}x_{ij})x_{ij} - y_{ij}(x_{ij}^2).$$

But $y_{ij}x_{ij} = 0$ from (19) and so

$$0 = (y_{ij}x_{ij})x_{ij} = 0.$$ (21)

Next, $0 = (x_{ij}, y_{ij}, e) + (y_{ij}, e, x_{ij}) + (e, x_{ij}, y_{ij}) = (x_{ij}, y_{ij}, e) + (y_{ij}, e, x_{ij})$ from (2) and (20). Expanding these associators we get

$$0 = (x_{ij}, y_{ij}) = 0.$$ (22)

Combining (21) and (22) and using (19) we obtain $0 = (x_{ij}y_{ij})x_{ij} = (x_{ij}, y_{ij}, x_{ij}) = -(x_{ij}, x_{ij}, y_{ij}) = -x_{ij}^2y_{ij}$. Then replacing $x_{ij}$ with $x_{ij} + z_{ij}$ in the last expression we get $(x_{ij}z_{ij} + z_{ij}x_{ij})y_{ij} = 0$. This result, combined with (22), yields $(x_{ij}z_{ij})y_{ij} = 0$. Hence $R_{ij} = 0$.

**Lemma 7.** $(R_i, R_{ij}^2) = 0$ where $i = 0$, 1 and $j = 1 - i$.

**Proof.** From Lemma 6 and (19), it suffices to consider commutators $(x, y)$ where $x$ belongs to $R_{ij} \cup R_{ij}$ and $y$ to $R_{ij}$. First, let $x_{ij}$, $y_{ij}$ belong to $R_{ij}$ and $z_{ji}$ to $R_{ji}$. Now $0 = (x_{ij}, y_{ij}, z_{ji}) + (x_{ij}, z_{ji}, y_{ij}) = (x_{ij}, z_{ji}, y_{ij})$ from (19). Then $0 = (x_{ij}, y_{ij}, z_{ji}) + (y_{ij}, z_{ji}, x_{ij}) + (z_{ji}, x_{ij}, y_{ij}) = (z_{ji}, x_{ij}, y_{ij}) = -z_{ji}(x_{ij}y_{ij})$ from (2) and (19). Hence $R_{ij}R_{ij}^2 = R_{ij}^2R_{ij} = 0$.

It remains to show that elements from $R_{ij}^2$ commute with elements of $R_{ij}$. To this end, let $x_{ij}$, $y_{ij}$ belong to $R_{ij}$ and $z_{ij}$ to $R_{ij}$. Then $0 = (x_{ij}, y_{ij}, z_{ij}) + (x_{ij}, z_{ij}, y_{ij}) = (x_{ij}y_{ij})z_{ij} - x_{ij}(z_{ij}y_{ij})$ from (19). Since the elements in $R_{ij}$ commute, (22), we have

$$0 = (x_{ij}y_{ij})z_{ij} = (y_{ij}x_{ij})z_{ij} = (z_{ij}x_{ij})y_{ij}.$$ (23)

Next, $0 = (z_{ij}, x_{ij}, y_{ij}) + (x_{ij}, y_{ij}, z_{ij}) + (y_{ij}, z_{ij}, x_{ij}) = (z_{ij}, x_{ij}, y_{ij}) + (x_{ij}y_{ij})z_{ij} - y_{ij}(z_{ij}x_{ij})$ from (2) and (19). Then from (19), (23) and the fact that elements from $R_{ij}$ commute with each other we conclude that

$$0 = (z_{ij}, x_{ij}, y_{ij}) = 0.$$ (24)

Expanding this associator and using (22) and (23) we get
0 = (z_{ij}x_{ij})y_{ij} - z_{ii}(x_{ij}y_{ij}) = (y_{ij}x_{ij})z_{ii} - z_{ii}(x_{ij}y_{ij}). Hence (R_{ii}, R_{ij}) = 0 and we conclude that (R, R_{ij}) = 0.

To prove that $R^2_{ij}$ where $i = 0, 1$ and $j = 1 - i$ are in the center of $R$ it remains to show that they are contained in the nucleus of $R$.

**Lemma 8.** $R^2_{ij}$ is in the center of $R$ when $i = 0, 1$ and $j = 1 - i$.

**Proof.** Consider associators of the form $(x_{kp}, y_{ij}z_{ij}, w_{mn})$ where $x_{kp}$ belongs to $R_{kp}$, $w_{mn}$ to $R_{mn}$ and $y_{ij}, z_{ij}$ to $R_{ij}$ and $k, p, m, n = 0, 1$.

We recall, first, that $R_{ij} + R_{ji} + R_{jj}$ annihilates $R^2_{ij}$ from both sides. Hence the above associator vanishes unless $k = p = i$ or $m = n = i$. Suppose that $k = p = i$. If $m = j$, $0 = (x_{ii}, y_{ij}z_{ij}, w_{jn})$ from (19) regardless of the value of $n$. Next, let $k = p = m = i$ and $n = j$. Then $0 = (x_{ii}, y_{ij}z_{ij}, w_{ij}) + (x_{ii}, w_{ij}, y_{ij}z_{ij}) = (x_{ii}, y_{ij}z_{ij}, w_{ij})$ from (19) and Lemma 7.

Elements from $R^2_{ij}$ belong to $U$ by Lemma 7. Hence, from (10), $(x_{kp}, y_{ij}z_{ij}, w_{mn}) = -(w_{mn}, y_{ij}z_{ij}, x_{kp})$. If we now assume that $m = n = i$ the argument of the last paragraph applies and we can conclude that unless $k = p = m = n = i$, the original associator vanishes.

We have left to consider associators of the form $(x_{ii}, y_{ij}z_{ij}, w_{ii})$. From (4) and (24) we obtain $0 = J(x_{ii}, w_{ii}, y_{ij}, z_{ij}) = (x_{ii}, w_{ii}, y_{ij}z_{ij}) + (x_{ii}, y_{ij}, w_{ii}z_{ij}) - (x_{ii}, w_{ii}, z_{ij})y_{ij} - (x_{ii}, y_{ij}, z_{ij})w_{ii} = (x_{ii}, w_{ii}, y_{ij}z_{ij}) - (x_{ii}, w_{ii}, z_{ij})y_{ij}$. However, $0 = (x_{ii}, w_{ii}, z_{ij}) + (x_{ii}, z_{ij}, w_{ii}) = (x_{ii}, w_{ii}, z_{ij})$ from (19) and so we have $0 = (x_{ii}, w_{ii}, y_{ij}z_{ij}) = -(x_{ii}, y_{ij}z_{ij}, w_{ii})$. Therefore, $R^2_{ij}$ is in the middle nucleus of $R$. It follows from (1) and (2) that whenever an element is contained in the middle nucleus of $R$ it is contained in the nucleus of $R$. Hence, $R^2_{ij}$ is in the nucleus of $R$.

**Lemma 9.** $R^2_{ij}$ is a trivial ideal of $R$, where $i = 0, 1$ and $j = 1 - i$.

**Proof.** $(R^2_{ij}, R) = 0$ and $(R_{ij} + R_{ji} + R_{jj})R^2_{ij} = 0$. Also, from (23), when $x_{ij}, y_{ij}$ belong to $R_{ij}$ and $z_{ij}$ to $R_{ii}$, $(x_{ij}y_{ij})z_{ij} = (z_{ij}y_{ij})x_{ij}$. Therefore, $R^2_{ij}$ is an ideal of $R$. But, from Lemmas 6 and 8, when $x_{ij}, y_{ij}, z_{ij}, w_{ij}$ belong to $R_{ij}$, $(x_{ij}y_{ij})(z_{ij}w_{ij}) = [(x_{ij}y_{ij})z_{ij}]w_{ij} = 0$. Hence, $R^2_{ij}$ is a trivial ideal of $R$.

**Corollary.** $R^2_{ij} = 0$ where $i = 0, 1$ and $j = 1 - i$.

**Proof.** Immediate from Lemma 9 and the fact that $R$ has no trivial ideals.

It is now clear that the submodules $R_{ij}$, where $i, j = 0, 1$, satisfy the same multiplicative relations as those for an associative ring with idempotent. Namely,

\[(25) \quad R_{ij}R_{kp} \subseteq \delta_{jk}R_{ip}\] where $i, j, k, p = 0, 1$ and $\delta_{jk}$ is the Kronecker delta.
As an immediate consequence of (25), by direct computation, \( e \) is contained in the nucleus of \( R \).

The next lemma is true for arbitrary rings of type \((-1, 1)\).

**Lemma 10.** Let \( y \) belong to the nucleus of \( R \) where \( R \) is a ring of type \((-1, 1)\). Then \((y, z)\) belongs to the nucleus of \( R \) for all \( z \) in \( R \).

**Proof.** Let \( x, w, z \) belong to \( R \). Then from (4), \( 0 = F(x, w, y, z) = (x, w, yz) + (x, y, wz) - (x, w, z)y = (x, w, yz) - (x, w, z)y \). Also, from (11), \( 0 = F(x, w, z, y) = (xw, z, y) - (x, wz, y) + (x, w, zy) - x(w, z, y) - (x, w, z)y = (x, w, zy) - (x, w, z)y \). Combining the above results, we conclude that \((x, w, yz) = (x, w, zy)\) and thus \((x, w, (y, z)) = 0\). Therefore, \((y, z)\) is in the right nucleus of \( R \). Finally, from (1) and (2), \((y, z)\) is contained in the nucleus of \( R \).

**Lemma 11.** The set \( B = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10} \) is an ideal in the nucleus of \( R \).

**Proof.** It was mentioned above that \( e \) is in the nucleus of \( R \). Let \( x_{ij} \) belong to \( R_{10} \). Then from Lemma 10, \((e, x_{ij})\) also belongs to the nucleus of \( R \). But \((e, x_{ij}) = \pm x_{ij}\) when \( i \neq j \) and so \( R_{10} \) and \( R_{01} \) are in the nucleus of \( R \). From this fact and (25), it is immediate that \( B \) is an ideal of \( R \) contained in the nucleus of \( R \).

We now make the additional assumption that \( R \) is a prime ring. A ring \( R \) is called prime if, whenever \( I \) and \( J \) are ideals in \( R \) such that \( IJ = 0 \), then either \( I = 0 \) or \( J = 0 \).

**Lemma 12.** Let \( R \) be an arbitrary nonassociative prime ring. Then \( R \) can contain no nonzero nuclear ideals.

**Proof.** Let \( A \) be an ideal in the nucleus of \( R \). Then if \( x, y, z, w \) belong to \( R \) and \( a \) to \( A \), we have \( 0 = F(a, x, y, z) = (ax, y, z) - (a, xy, z) + (a, x, yz) - a(x, y, z) - (a, x, y)z - a(x, y, z) \). Further, \( a \[(x, y, z)w] = [a(x, y, z)]w = 0 \). But finite sums of elements of the form \((R, R, R)\) and \((R, R, R)R\) form a 2-sided ideal in an arbitrary ring. Hence \( A \) annihilates an ideal of \( R \) containing all associators of \( R \). Since \( R \) is prime and not-associative, \( A = 0 \).

It is now clear that \( B = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10} = 0 \) and \( R \) assumes the form \( R = R_{11} + R_{00} \) unless \( R \) is associative. But then \( R_{11} \) and \( R_{00} \) become orthogonal ideals of \( R \). Since \( R \) is prime this means that either \( R_{11} = 0 \) or \( R_{00} = 0 \). However, \( e \neq 0 \) belongs to \( R_{11} \). Thus \( R_{00} = 0 \). If this is the case, \( R = R_{11} \) and \( e \) is the identity of \( R \), contrary to our assumption that \( e \neq 1 \). Hence, \( R \) must be associative.

**Theorem 2.** Let \( R \) be a prime ring of type \((-1, 1)\) with characteristic
prime to 6. If $R$ has an idempotent $e \neq 0, 1$, then $R$ is associative.

It should be noted that an arbitrary primitive ring is also prime [9]. Hence, by constructing a suitable radical, the results of this paper could be extended to semisimple rings.

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