

A NOTE ON CARTAN'S THEOREMS A AND B

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In this short note we show that Cartan's Theorem A is an easy consequence of Cartan's Theorem B (for both reduced and unreduced complex spaces).

Assume that Theorem B is true. Suppose \mathcal{F} is a coherent analytic sheaf on a Stein space (X, \mathcal{O}) and $x \in X$. Let \mathcal{g} be the sheaf of germs of holomorphic functions vanishing at x . We have an epimorphism $\phi: \mathcal{O}_x^p \rightarrow \mathcal{F}_x$. Let $e_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathcal{O}_x^p$, (the 1 is in the k th position, $1 \leq k \leq p$). Consider the exact sequence of sheaf-homomorphisms

$$(*) \quad 0 \rightarrow \mathcal{g}\mathcal{F} \rightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{F}/\mathcal{g}\mathcal{F} \rightarrow 0.$$

Since $(\mathcal{F}/\mathcal{g}\mathcal{F})_y = 0$ for $y \neq x$, $(\psi \circ \phi)(e_k) \in (\mathcal{F}/\mathcal{g}\mathcal{F})_x$ define sections $s_k \in \Gamma(X, \mathcal{F}/\mathcal{g}\mathcal{F})$, $1 \leq k \leq p$. Cartan's Theorem B implies that $H^1(X, \mathcal{g}\mathcal{F}) = 0$. From the exact cohomology sequence of (*) we conclude that ψ induces an epimorphism $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}/\mathcal{g}\mathcal{F})$. There exist $f_k \in \Gamma(X, \mathcal{F})$ such that $\psi(f_k) = s_k$, $1 \leq k \leq p$. We claim that $(f_1)_x, \dots, (f_p)_x$ generate \mathcal{F}_x . Take $u \in \mathcal{F}_x$. Then $\phi(v) = u$ for some $v \in \mathcal{O}_x^p$. $\psi(\phi(e_k) - (f_k)_x) = 0$ implies that $\phi(e_k) - (f_k)_x \in \mathcal{g}_x \mathcal{F}_x = \mathcal{g}_x \phi(\mathcal{O}_x^p) = \phi(\mathcal{g}_x \mathcal{O}_x^p)$. $\phi(e_k) - (f_k)_x = \phi(g_k)$ for some $g_k = (g_{k1}, \dots, g_{kp}) \in \mathcal{g}_x \mathcal{O}_x^p$, $1 \leq k \leq p$. $\det(\delta_{kl} - g_{kl})_{1 \leq k \leq p, 1 \leq l \leq p}$ is a unit in \mathcal{O}_x , where $\delta_{kl} = 1 \in \mathcal{O}_x$ if $k = l$, and $\delta_{kl} = 0 \in \mathcal{O}_x$ if $k \neq l$. $e_1 - g_1, \dots, e_p - g_p$ generate \mathcal{O}_x^p . $v = \sum_{k=1}^p \lambda_k (e_k - g_k)$ for some $\lambda_k \in \mathcal{O}_x$, $1 \leq k \leq p$. $u = \phi(v) = \sum_{k=1}^p \lambda_k \phi(e_k - g_k) = \sum_{k=1}^p \lambda_k (f_k)_x$. Theorem A is proved. Moreover, we have proved that the least number of global sections required to generate a given stalk is the same as the least number of elements of the stalk to generate the stalk. This gives us the following:

COROLLARY. *Suppose Z is a subvariety in an open subset (G, \mathcal{O}) of a complex number space. The set of points $\sigma(Z)$ of Z , at which Z is not an algebraic complete intersection, is a subvariety.*

PROOF. Without loss of generality we can assume that G is Stein. Let the ideal-sheaf of Z be \mathcal{A} . If Z is of pure codimension p , then

$$\sigma(Z) = \bigcap \left\{ \left\{ z \mid z \in Z, \left(\sum_{i=1}^p \mathcal{A}f_i \right)_z \neq \mathcal{A}_z \right\} \mid f_1, \dots, f_p \in \Gamma(G, \mathcal{A}) \right\}$$

Received by the editors March 13, 1967.

and hence is a subvariety. If $Z = \bigcup_q Z^q$ is the decomposition into pure dimensional subvarieties, where $\text{codim } Z^q = q$, then since at points not in $\sigma(Z)$, Z is pure dimensional, $\sigma(Z) = (\bigcup_q \sigma(Z^q)) \cup (\bigcup_{q \neq r} (Z^q \cap Z^r))$. Q.E.D.

REFERENCE

1. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables* Prentice-Hall, Englewood Cliffs, N. J., 1965.

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ON THE MAGNITUDE OF $x^n - 1$ IN A NORMED ALGEBRA

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In their work on topological near-rings, Beidleman and Cox¹ had occasion to prove the following theorem: If A is a linear transformation on C^n (with a norm $\|\cdot\|$) and if $\|A^n - I\| \leq \alpha$ for all n with $0 \leq \alpha < 1$, then $A = I$. Their proof consists in noting that $A - I$ is nilpotent and then examining the Jordan form.

It is certainly natural to inquire whether this theorem extends to the case where A is a bounded linear transformation on a normed linear space, or is in a normed algebra. Below we give an extremely short proof that the theorem does extend.

THEOREM. *If X is a normed algebra with 1 and $\alpha_n = \|x^n - 1\|$ satisfies $\alpha_n = o(n)$ and $\liminf n^{-1}(\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}) = \beta < 1$, then $x = 1$. In particular $x = 1$ if $\alpha_n \leq \alpha < 1$.*

PROOF. Write

$$\begin{aligned} (x - 1) &= \frac{x^n - 1}{n} + (x - 1) \left[1 - \frac{1 + x + \dots + x^{n-1}}{n} \right] \\ &= \frac{(x^n - 1)}{n} + \frac{(x - 1)}{n} [(1 - x) + (1 - x^2) + \dots + (1 - x^{n-1})]. \end{aligned}$$

Hence $\|x - 1\| \leq [\alpha_n + \|x - 1\|(\alpha_1 + \alpha_2 + \dots + \alpha_{n-1})]/n$. Letting n tend to ∞ through a suitable sequence, we get $\|x - 1\| \leq \beta \|x - 1\|$ giving the result.

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Received by the editors March 23, 1967.

¹ Private communication.