AN APPROXIMATION METHOD FOR WIENER INTEGRALS OF CERTAIN UNBOUNDED FUNCTIONALS

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Introduction. In Cameron's paper [1] there appears a "rectangle formula" (there denoted by Theorem 1) by means of which the Wiener integral of $F[x(\cdot)]$, defined on the space of continuous functions on $[0, 1]$ such that $x(0)=0$, can be approximated by means of an n-fold Riemann integral provided F is sufficiently smooth and dominated by a suitable integrable functional. The formula employs a particular C.O.N. set of functions $\{\alpha_i(s)\}$[1] (the odd harmonic cosine functions.) A generalization of this formula appears in Cameron's unpublished notes and appears with proof in the author's [2], (there denoted by Theorem 2). The generalization allows for the use of an arbitrary C.O.N. set $\{\alpha_i(s)\}$ of B.V. but suffers the defect that it is not known to hold except for bounded functionals. It is the purpose of this paper to show that if the $\{\alpha_i(s)\}$'s of this generalization are restricted to certain Sturm-Liouville sets of functions (among which sets is included the odd harmonic cosine functions), then the restricted generalization is applicable to suitably dominated (not necessarily bounded) functionals.

Let $w(s)>0$ have continuous fourth derivative on $[0, 1]$ and let $\{\lambda_i\}$ and $\{\alpha_i(s)\}$ be the sets of characteristic numbers and normalized functions of the Sturm-Liouville problem: $\{[1/w(s)]\alpha'(s)\}' +\lambda\alpha(s)=0$, $\alpha'(0)=\alpha(1)=0$ on $[0, 1]$. Let C be the space of continuous functions $x(s)$ on $[0, 1]$ such that $x(0)=0$, let $\beta_i(s)=\int_0^s\alpha_i(t)dt$ and for $x(\cdot)\in C$ let $c_i=\int_0^1\alpha_i(s)dx(s)$: $i=1, 2, 3, \cdots$ (c_i is of course a functional of $x(\cdot)$). The main result of the paper is the following.

THEOREM. *Let $F[x(\cdot)]$ be continuous in the Hilbert topology on C and let*

$$|F[x(\cdot)]| \leqq H\left(\int_0^1 w(t)x^2(t)dt\right) \quad on\ C,$$

where $H(u)$ is monotonically increasing on $[0, 1]$ and

$$\int_C H\left(\int_0^1 w(t)x^2(t)dt\right)dx < \infty.$$

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[1] The β's of Cameron's paper [1] correspond to the α's of this paper, and the α's of Cameron's paper are constant multiples of the β's of this paper.

Then

$$\lim_{n\to\infty} \int_C F\left[\sum_{i=1}^{n} c_i\beta_i(\cdot)\right]dx = \int_C F[x(\cdot)]dx.$$

It can easily be verified by letting $w(t)\equiv 1$ that Cameron's original rectangle formula is a special case of the theorem.

In §1 will be given five lemmas and by means of them a proof of the theorem in §2.

1. Five lemmas.

LEMMA 1.1. *If* $\{f_i(s)\}$ *is a complete set of functions on* $[0, 1]$ *then so also is* $\{\int_0^s f_i(t)dt\}$.

PROOF. If for $g(s)\in L_2[0, 1]$,

$$\int_0^1 g(s)\int_0^s f_i(t)dtds = 0, \qquad i = 1, 2, 3, \cdots,$$

then by interchanging the order of integration there follows

$$\int_0^1\int_t^1 g(s)dsf_i(t)dt = 0, \qquad i = 1, 2, 3, \cdots,$$

so that $\int_t^1 g(s)ds$, and hence $g(s)$ itself, equals zero almost everywhere.

LEMMA 1.2. *If* $\{f_i(s)\}$ *is a complete set of functions on* $[0, 1]$ *and if* $h(s)$ *is continuous and* $h(s)>0$ *on* $[0, 1]$ *then* $\{h(s)f_i(s)\}$ *is a complete set of functions on* $[0, 1]$.

PROOF. There will be shown that any $g(s)\in L_2[0, 1]$ can be approximated arbitrarily closely in the L_2 sense by a linear combination of the $h(s)\, f_i(s)$'s. Let a_i, $i=1, 2, 3, \cdots, n$, be unspecified constants.

$$\begin{aligned}
&\int_0^1\left[g(s) - \sum_{i=1}^{n} a_ih(s)f_i(s)\right]^2 ds \\
&\qquad = \int_0^1 [h(s)]^2\left[g(s)/h(s) - \sum_{i=1}^{n} a_if_i(s)\right]^2 ds \\
&\qquad \leqq \left[\max_{0\leqq s\leqq 1} h(s)\right]^2\int_0^1\left[g(s)/h(s) - \sum_{i=1}^{n} a_if_i(s)\right]^2 ds.
\end{aligned} \tag{1.1}$$

Now since $g(s)/h(s)\in L_2[0, 1]$, suitable n and a_i's can be found to make the integral in the last member of (1.1) arbitrarily small. This completes the proof of the lemma.

LEMMA 1.3. *Let* $w(s)>0$ *be continuous on* $[0, 1]$. *If* λ_i *and* $\alpha_i(s)$ *are* *the characteristic numbers and normal functions respectively of the Sturm-Liouville problem:* $\{[1/w(s)]\alpha'(s)\}'+\lambda\alpha(s)=0,\ \alpha'(0)=\alpha(1)=0$ *on* $[0, 1]$, *then* $\{\sqrt{[\lambda_i w(s)]}\beta_i(s)\}$ *is a complete orthonormal set of functions on* $[0, 1]$.

PROOF. From the theory of Sturm-Liouville problems e.g. Ince [**3**, 241] it is known that the characteristic numbers occur with multiplicity one and that to each characteristic number corresponds just one (independent) characteristic function. Also, a result similar to [**3**, 237–238, 273–276] is that the normalized characteristic functions constitute a C.O.N. set of functions. That all the characteristic numbers are positive will now be shown. One integration of the Sturm-Liouville equation yields

$$\beta_i(s) = -\alpha_i'(s)/[\lambda_i w(s)]. \tag{1.2}$$

If both sides of (1.2) are multiplied by $\lambda_i w(s)\beta_i(s)$ and integrated there follows

$$\begin{aligned}\lambda_i\int_0^1 w(s)\beta_i^2(s)ds &= -\int_0^1 \beta_i(s)\alpha_i'(s)ds\\ &= -\beta_i(s)\alpha_i(s)\Big|_0^1 + \int_0^1 \alpha_i^2(s)ds = 1,\end{aligned}$$

(because $\beta_i(0)=\alpha_i(1)=0$) and thus λ_i is positive.

Since $\{\alpha_i(s)\}$ is C.O.N. there follows from Lemmas 1.1 and 1.2 that $\{\sqrt{[\lambda_i w(s)]}\beta_i(s)\}$ is a complete set of functions.

There need only be verified that the latter is an orthonormal set. But from (1.2) follows

$$\begin{aligned}&\int_0^1 \sqrt{[\lambda_i w(s)]}\beta_i(s)\ \sqrt{[\lambda_j w(s)]}\beta_j(s)ds\\ &= -(\lambda_i/\lambda_j)^{1/2}\int_0^1 \beta_i(s)\alpha_j'(s)ds\\ &= -(\lambda_i/\lambda_j)^{1/2}\left[\beta_i(s)\alpha_j(s)\Big|_0^1 - \int_0^1 \alpha_j(s)\alpha_i(s)ds\right]\\ &= (\lambda_i/\lambda_j)^{1/2}\int_0^1 \alpha_i(s)\alpha_j(s)ds\\ &\qquad\qquad\qquad\qquad (\text{because } \beta_i(0) = \alpha_i(1) = 0)\\ &= \delta_{ij}\end{aligned}$$

which completes the required verification.

LEMMA 1.4.

$$\int_0^1 w(t)\left[\sum_{i=1}^n c_i\beta_i(t)\right]^2 dt = \sum_{i=1}^n c_i^2/\lambda_i \leqq \int_0^1 w(t)x^2(t)dt, \tag{1.3}$$

$$\lim_{n\to\infty}\int_0^1 w(t)\left[\sum_{i=1}^n c_i\beta_i(t)\right]^2 dt = \sum_{i=1}^\infty c_i^2/\lambda_i = \int_0^1 w(t)x^2(t)dt. \tag{1.4}$$

PROOF. (1.3) and (1.4) are respectively Bessel's inequality and Parseval's equation for the development of $[w(t)]^{1/2}x(t)$ with respect to the C.O.N. set $\{[\lambda_i w(t)]^{1/2}\beta_i(t)\}$. The verification of this assertion follows.

$$\int_0^1 [w(t)]^{1/2}x(t)[\lambda_i w(t)]^{1/2}\beta_i(t)dt$$

$$= -\int_0^1 [w(t)]^{1/2}x(t)[\lambda_i w(t)]^{1/2}\alpha_i'(t)/[\lambda_i w(t)]dt$$

(because of (1.2))

$$= -\int_0^1 x(t)\alpha_i'(t)/(\lambda_i)^{1/2}dt$$

$$= -x(t)\alpha_i(t)/(\lambda_i)^{1/2}\Big|_0^1 + \int_0^1 \alpha_i(t)dx(t)/(\lambda_i)^{1/2}$$

$$= c_i/(\lambda_i)^{1/2}$$

(because $x(0) = \alpha_i(1) = 0$).

LEMMA 1.5.

$$\lim_{n\to\infty}\int_0^1\left[x(t) - \sum_{i=1}^n c_i\beta_i(t)\right]^2 dt = 0.$$

PROOF.

$$\int_0^1\left[\min_{0\leqq t\leqq 1} w(t)\right]\left[x(t) - \sum_{i=1}^n c_i\beta_i(t)\right]^2 dt$$

$$\leqq \int_0^1 w(t)\left[x(t) - \sum_{i=1}^n c_i\beta_i(t)\right]^2 dt$$

and the limit of the right side of the inequality is, by Lemma 1.4, equal to zero. Since $[\min_{0\leq t\leq 1} w(t)]$ is a constant greater than zero the proof is complete.

2. **The proof of the theorem.** Now will be given the proof of the theorem stated in the introduction. First there is observed that because of Lemma 1.4 and the assumption on F

$$\left|F\left[\sum_{i=1}^{n} c_i\beta_i(\cdot)\right]\right| \leqq H\left(\int_0^1 w(t)\left[\sum_{i=1}^{n} c_i\beta_i(t)\right]^2 dt\right)$$
$$\leqq H\left(\int_0^1 w(t)x^2(t)dt\right).$$

Also, from Lemma 1.5 and the continuity of F in the Hilbert topology

$$\lim_{n\to\infty} F\left[\sum_{i=1}^{n} c_i\beta_i(\cdot)\right] = F[x(\cdot)].$$

Lebesgue's dominated convergence theorem completes the proof.

Finally there will be shown that there is a nontrivial $H(u)$ satisfying the hypotheses of the theorem. For any $k<\lambda_0$ (again note $\lambda_0>0$ so that k can be chosen positive) $\exp(ku)$ will serve as $H(u)$. To prove this there is observed that Wiener's formula for functions of n linear functionals, viz.

$$\int_C f(c_1, \cdots, c_n)dx$$
$$= \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \pi^{-n/2} \exp(-\xi_1^2 \cdots - \xi_n^2) f(\xi_1, \cdots, \xi_n) d\xi_1 \cdots d\xi_n$$

yields that

$$\int_C \exp\left(k\int_0^1 w(t)\left[\sum_{i=1}^{n} c_i\beta_i(t)\right]^2 dt\right) dx$$
$$= \int_C \exp\left(k\sum_{i=1}^{n} c_i^2/\lambda_i\right) dx$$
$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \pi^{-1/2} \exp(-\xi_i^2 + k\xi_i^2/\lambda_i) d\xi_i$$
$$= \prod_{i=1}^{n} \{1/\sqrt{[1 - k/\lambda_i]}\}.$$

The estimate given for λ_i in [**3**, 272] ensures that the infinite product converges. An appeal to Lebesgue's monotone convergence theorem completes the establishment of the result.

References

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REMARKS ON SOME CONVERGENCE CONDITIONS FOR CONTINUED FRACTIONS

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In [**4**] Farinha proved the following theorem concerning the continued fraction

$$\frac{1}{1}+\frac{a_1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\cdots. \tag{1}$$

Theorem. *Suppose the a_k are functions of a complex variable defined in a region D. If the a_k satisfy at each point of D the conditions*

(i) *no term of $\{a_n\}$ is zero, but* $\lim_n a_n=0$,

(ii) $|a_1|\leqq\alpha$ *and* $|1+a_1|\geqq|a_1|+\mu$, *where α and μ are positive constants, and*

(iii) $|1+a_n+a_{n+1}|\geqq 2|a_i|$, $i=n,\ n+1$; $n=1,\ 2,\ 3,\ \cdots$, *then the continued fraction* (1) *converges uniformly over D and the modulus of its value does not exceed the smaller of the numbers $3/2$ and* $(\alpha+\mu)/\mu^2$.

Convergence of (1) under Farinha's three conditions follows from a theorem of Scott and Wall [**6**, Theorem 3.4].

In this note we show that (iii) alone is sufficient to give convergence of the continued fraction (Theorem 1), and, using the basic idea involved in the proof, we extend a theorem of the author [**3**, Theorem A].

Theorem 1. *If for each positive integer n, $|1+a_n+a_{n+1}|\geqq 2|a_i|$, $i=n,\ n+1$, then the continued fraction* (1) *converges.*

Proof. From the hypothesis and the triangle inequality we can

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