AN APPROXIMATION METHOD FOR
WIENER INTEGRALS OF CERTAIN
UNBOUNDED FUNCTIONALS

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Introduction. In Cameron’s paper [1] there appears a “rectangle
formula” (there denoted by Theorem 1) by means of which the
Wiener integral of $F[x(\cdot)]$, defined on the space of continuous func-
tions on $[0, 1]$ such that $x(0) = 0$, can be approximated by means of an
$n$-fold Riemann integral provided $F$ is sufficiently smooth and dom-
inated by a suitable integrable functional. The formula employs a
particular C.O.N. set of functions $\{\alpha_i(s)\}$ (the odd harmonic cosine
functions.) A generalization of this formula appears in Cameron’s
unpublished notes and appears with proof in the author’s [2],
(there denoted by Theorem 2). The generalization allows for the use
of an arbitrary C.O.N. set $\{\alpha_i(s)\}$ of B.V. but suffers the defect that
it is not known to hold except for bounded functionals. It is the pur-
pose of this paper to show that if the $\{\alpha_i(s)\}$’s of this generalization
are restricted to certain Sturm-Liouville sets of functions (among
which sets is included the odd harmonic cosine functions), then the
restricted generalization is applicable to suitably dominated (not
necessarily bounded) functionals.

Let $w(s) > 0$ have continuous fourth derivative on $[0, 1]$ and let
$\{\lambda_i\}$ and $\{\alpha_i(s)\}$ be the sets of characteristic numbers and normal-
ized functions of the Sturm-Liouville problem: $\left(1/w(s)\right)\alpha''(s) + \lambda \alpha(s) = 0$, $\alpha'(0) = \alpha(1) = 0$ on $[0, 1]$. Let $C$ be the space of continu-
sous functions $x(s)$ on $[0, 1]$ such that $x(0) = 0$, let $\beta_i(s) = \int_0^s \alpha_i(t) dt$
and for $x(\cdot) \in C$ let $c_i = \int_0^1 \alpha_i(s) dx(s)$: $i = 1, 2, 3, \cdots$ ($c_i$ is of course a
functional of $x(\cdot)$). The main result of the paper is the following.

Theorem. Let $F[x(\cdot)]$ be continuous in the Hilbert topology on $C$
and let

$$|F[x(\cdot)]| \leq H \left( \int_0^1 w(t) x^2(t) dt \right) \quad \text{on } C,$$

where $H$ is monotonically increasing on $[0, 1]$ and

$$\int_c H \left( \int_0^1 w(t) x^2(t) dt \right) dx < \infty.$$

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1 The $\beta$’s of Cameron’s paper [1] correspond to the $\alpha$’s of this paper, and the $\alpha$’s
of Cameron’s paper are constant multiples of the $\beta$’s of this paper.
Then
\[
\lim_{n \to \infty} \int_0^1 F \left[ \sum_{i=1}^n c_i \delta_i(\cdot) \right] dx = \int_0^1 F [x(\cdot)] dx.
\]

It can easily be verified by letting \( w(t) \equiv 1 \) that Cameron's original rectangle formula is a special case of the theorem.

In §1 will be given five lemmas and by means of them a proof of the theorem in §2.

1. Five lemmas.

**Lemma 1.1.** If \( \{f_i(s)\} \) is a complete set of functions on \([0, 1]\) then so also is \( \{\int_0^s f_i(t) dt\} \).

**Proof.** If for \( g(s) \in L^2[0, 1] \),
\[
\int_0^1 g(s) \int_0^s f_i(t) dt ds = 0, \quad i = 1, 2, 3, \ldots,
\]
then by interchanging the order of integration there follows
\[
\int_0^1 \int_0^1 g(s) df_i(t) dt ds = 0, \quad i = 1, 2, 3, \ldots,
\]
so that \( \int_0^1 g(s) ds \), and hence \( g(s) \) itself, equals zero almost everywhere.

**Lemma 1.2.** If \( \{f_i(s)\} \) is a complete set of functions on \([0, 1]\) and \( h(s) \) is continuous and \( h(s) > 0 \) on \([0, 1]\) then \( \{h(s)f_i(s)\} \) is a complete set of functions on \([0, 1]\).

**Proof.** There will be shown that any \( g(s) \in L^2[0, 1] \) can be approximated arbitrarily closely in the \( L^2 \) sense by a linear combination of the \( h(s)f_i(s) \)'s. Let \( a_i, i = 1, 2, 3, \ldots, n, \) be unspecified constants.

\[
\int_0^1 \left[ g(s) - \sum_{i=1}^n a_i h(s)f_i(s) \right]^2 ds
\]
\[
= \int_0^1 \left[ h(s) \right]^2 \left[ g(s)/h(s) - \sum_{i=1}^n a_i f_i(s) \right]^2 ds
\]
\[
\leq \left[ \max_{0 \leq s \leq 1} h(s) \right]^2 \int_0^1 \left[ g(s)/h(s) - \sum_{i=1}^n a_i f_i(s) \right]^2 ds.
\]

Now since \( g(s)/h(s) \in L^2[0, 1] \), suitable \( n \) and \( a_i \)'s can be found to make the integral in the last member of (1.1) arbitrarily small. This completes the proof of the lemma.

**Lemma 1.3.** Let \( w(s) > 0 \) be continuous on \([0, 1]\). If \( \lambda_i \) and \( \alpha_i(s) \) are
the characteristic numbers and normal functions respectively of the Sturm-Liouville problem: \( \{ 1/w(s) \alpha'(s) \} ' + \lambda \alpha(s) = 0, \alpha'(0) = \alpha(1) = 0 \) on \([0,1]\), then \( \{ \sqrt{\lambda} w(s) \beta_i(s) \} \) is a complete orthonormal set of functions on \([0,1]\).

**Proof.** From the theory of Sturm-Liouville problems e.g. Ince [3, 241] it is known that the characteristic numbers occur with multiplicity one and that to each characteristic number corresponds just one (independent) characteristic function. Also, a result similar to [3, 237–238, 273–276] is that the normalized characteristic functions constitute a C.O.N. set of functions. That all the characteristic numbers are positive will now be shown. One integration of the Sturm-Liouville equation yields

\[
\beta_i(s) = - \alpha'_i(s)/[\lambda_i w(s)].
\]

If both sides of (1.2) are multiplied by \( \lambda_i w(s) \beta_i(s) \) and integrated there follows

\[
\lambda_i \int_0^1 w(s) \beta_i^2(s) ds = - \int_0^1 \beta_i(s) \alpha'_i(s) ds
\]

\[
= - \beta_i(s) \alpha_i(s) \bigg|_0^1 + \int_0^1 \alpha_i(s) ds = 1,
\]

(because \( \beta_i(0) = \alpha_i(1) = 0 \)) and thus \( \lambda_i \) is positive.

Since \( \{ \alpha_i(s) \} \) is C.O.N. there follows from Lemmas 1.1 and 1.2 that \( \{ \sqrt{\lambda_i w(s)} \beta_i(s) \} \) is a complete set of functions.

There need only be verified that the latter is an orthonormal set. But from (1.2) follows

\[
\int_0^1 \sqrt{\lambda_i w(s)} \beta_i(s) \sqrt{\lambda_j w(s)} \beta_j(s) ds
\]

\[
= - (\lambda_i/\lambda_j)^{1/2} \int_0^1 \beta_i(s) \alpha'_j(s) ds
\]

\[
= - (\lambda_i/\lambda_j)^{1/2} \left[ \beta_i(s) \alpha_j(s) \bigg|_0^1 - \int_0^1 \alpha_j(s) \alpha_i(s) ds \right]
\]

\[
= (\lambda_i/\lambda_j)^{1/2} \int_0^1 \alpha_i(s) \alpha_j(s) ds
\]

\[
= \delta_{ij} \quad (\text{because } \beta_i(0) = \alpha_i(1) = 0)
\]

which completes the required verification.
**Lemma 1.4.**

(1.3) \( \int_0^1 w(t) \left[ \sum_{i=1}^n c_i \beta_i(t) \right]^2 dt = \sum_{i=1}^n c_i^2 / \lambda_i \leq \int_0^1 w(t) x^2(t) dt, \)

(1.4) \( \lim_{n \to \infty} \int_0^1 w(t) \left[ \sum_{i=1}^n c_i \beta_i(t) \right]^2 dt = \sum_{i=1}^\infty c_i^2 / \lambda_i = \int_0^1 w(t) x^2(t) dt. \)

**Proof.** (1.3) and (1.4) are respectively Bessel's inequality and Parseval's equation for the development of \( [w(t)]^{1/2} x(t) \) with respect to the C.O.N. set \( \{ [\lambda_i w(t)]^{1/2} \beta_i(t) \} \). The verification of this assertion follows.

\[
\int_0^1 [w(t)]^{1/2} x(t) [\lambda_i w(t)]^{1/2} \beta_i(t) dt = - \int_0^1 [w(t)]^{1/2} x(t) [\lambda_i w(t)]^{1/2} \alpha_i(t) / [\lambda_i w(t)] dt
\]

(because of (1.2))

\[
= - \int_0^1 x(t) \alpha_i(t) / (\lambda_i)^{1/2} dt
\]

\[
= - x(t) \alpha_i(t) / (\lambda_i)^{1/2} \bigg|_0^1 + \int_0^1 \alpha_i(t) dx(t) / (\lambda_i)^{1/2}
\]

\[
= c_i / (\lambda_i)^{1/2}
\]

(because \( x(0) = \alpha_i(1) = 0 \)).

**Lemma 1.5.**

\[
\lim_{n \to \infty} \int_0^1 \left[ x(t) - \sum_{i=1}^n c_i \beta_i(t) \right]^2 dt = 0.
\]

**Proof.**

\[
\int_0^1 \left[ \min_{0 \leq t \leq 1} w(t) \right] \left[ x(t) - \sum_{i=1}^n c_i \beta_i(t) \right]^2 dt \leq \int_0^1 w(t) \left[ x(t) - \sum_{i=1}^n c_i \beta_i(t) \right]^2 dt
\]

and the limit of the right side of the inequality is, by Lemma 1.4, equal to zero. Since \( \min_{0 \leq t \leq 1} w(t) \) is a constant greater than zero the proof is complete.
2. **The proof of the theorem.** Now will be given the proof of the theorem stated in the introduction. First there is observed that because of Lemma 1.4 and the assumption on $F$

$$\left| F \left[ \sum_{i=1}^{n} c_i \beta_i(\cdot) \right] \right| \leq H \left( \int_{0}^{1} w(t) \left[ \sum_{i=1}^{n} c_i \beta_i(t) \right]^2 dt \right) \leq H \left( \int_{0}^{1} w(t) x^2(t) dt \right).$$

Also, from Lemma 1.5 and the continuity of $F$ in the Hilbert topology

$$\lim_{n \to \infty} F \left[ \sum_{i=1}^{n} c_i \beta_i(\cdot) \right] = F[x(\cdot)].$$

Lebesgue's dominated convergence theorem completes the proof.

Finally there will be shown that there is a nontrivial $H(u)$ satisfying the hypotheses of the theorem. For any $k < \lambda_0$ (again note $\lambda_0 > 0$ so that $k$ can be chosen positive) $\exp(ku)$ will serve as $H(u)$. To prove this there is observed that Wiener's formula for functions of $n$ linear functionals, viz.

$$\int_{c} f(c_1, \ldots, c_n) dx = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \pi^{-n/2} \exp(-\frac{\xi_1^2}{2} - \cdots - \frac{\xi_n^2}{2}) f(\xi_1, \ldots, \xi_n) d\xi_1 \cdots d\xi_n$$

yields that

$$\int_{c} \exp \left( k \int_{0}^{1} w(t) \left[ \sum_{i=1}^{n} c_i \beta_i(t) \right]^2 dt \right) dx = \int_{c} \exp \left( k \sum_{i=1}^{n} \frac{c_i^2}{\lambda_i} \right) dx = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \pi^{-1/2} \exp(-\frac{\xi_i^2}{2} + k \frac{\xi_i^2}{\lambda_i}) d\xi_i = \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{1 - k/\lambda_i}} \right\}. $$

The estimate given for $\lambda_i$ in [3, 272] ensures that the infinite product converges. An appeal to Lebesgue's monotone convergence theorem completes the establishment of the result.
REMARKS ON SOME CONVERGENCE CONDITIONS
FOR CONTINUED FRACTIONS

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In [4] Farinha proved the following theorem concerning the continued fraction

\[
\frac{1}{1 + \frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{\cdots}}}}
\]

Theorem. Suppose the \(a_k\) are functions of a complex variable defined in a region \(D\). If the \(a_k\) satisfy at each point of \(D\) the conditions

(i) no term of \(\{a_n\}\) is zero, but \(\lim_{n \to \infty} a_n = 0\),

(ii) \(|a_1| \leq \alpha\) and \(|1 + a_1| \geq |a_1| + \mu\), where \(\alpha\) and \(\mu\) are positive constants, and

(iii) \(|1 + a_n + a_{n+1}| \geq 2|a_i|, i = n, n+1; n = 1, 2, 3, \ldots\), then the continued fraction (1) converges uniformly over \(D\) and the modulus of its value does not exceed the smaller of the numbers \(3/2\) and \((\alpha + \mu)/\mu^2\).

Convergence of (1) under Farinha's three conditions follows from a theorem of Scott and Wall [6, Theorem 3.4].

In this note we show that (iii) alone is sufficient to give convergence of the continued fraction (Theorem 1), and, using the basic idea involved in the proof, we extend a theorem of the author [3, Theorem A].

Theorem 1. If for each positive integer \(n\), \(|1 + a_n + a_{n+1}| \geq 2|a_i|, i = n, n+1\), then the continued fraction (1) converges.

Proof. From the hypothesis and the triangle inequality we can

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References


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