A GENERALIZATION OF THE GÖLLNITZ-GORDON PARTITION THEOREMS

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1. Introduction. Among the most striking results in the theory of partitions are the Rogers-Ramanujan identities [**9**, p. 291]. These may be stated combinatorially as follows.

(1.1) The number of partitions of n with minimal difference 2 is equal to the number of partitions of n into parts of the forms $5m+1$ and $5m+4$.

(1.2) The number of partitions of n into parts not less than 2, and with minimal difference 2, is equal to the number of partitions of n into parts of the forms $5m+2$ and $5m+3$.

In 1926, I. J. Schur proved the following theorem which is similar to the above results [**10**].

(1.3) The number of partitions of n of the form $n=b_1+\cdots+b_s$, where $b_i-b_{i+1}\geqq 3$, and $b_i-b_{i+1}>3$ if $3\mid b_i$, is equal to the number of partitions of n into parts of the forms $6m+1$ and $6m+5$.

However, in 1948 H. L. Alder shut the door on further generalizations in this direction by proving the following three theorems [**1**]. Here $q_{d,m}(n)$ is the number of partitions of n into parts differing by at least d, each part being greater than or equal to m.

(1.4) Let S be any fixed set of positive integers, then $q_{d,m}(n)$ is not always equal to the number of partitions of n into parts taken from S if $d>2$.

(1.5) Let S be any fixed set of positive integers, then $q_{d,m}(n)$ is not equal to the number of partitions of n into distinct parts taken from S if $d>1$.

(1.6) Let S be any fixed set of positive integers, then the number of partitions of n into parts differing by at least d and where no consecutive multiples of d appear is not equal to the number of partitions of n into parts taken from S if $d>3$.

The case of (1.6) in which $d=2$ was treated independently by H. Göllnitz [**6**, p. 33–34] in 1960 and by B. Gordon [**8**, p. 741] in 1965. They proved the following two identities.

(1.7) The number of partitions of any positive integer n into parts -1, 4, 7 (mod 8) is equal to the number of partitions of the form $n=b_1+\cdots+b_s$, where $b_i-b_{i+1}\geqq 2$, and $b_i-b_{i+1}>2$ if b_i is even.

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(1.8) The number of partitions of any positive integer n into parts $\equiv 3, 4, 5 \pmod 8$ is equal to the number of partitions of the form $n=b_1+\cdots+b_s$ satisfying $b_s \geqq 3$ in addition to the inequalities of (1.7).

A different form of generalization of the Rogers-Ramanujan identities was discovered in 1961 by B. Gordon [7]. He proved the following result.

(1.9) Let a and k be integers with $0<a\leqq k$. Let $A_{k,a}(n)$ denote the number of partitions of n into parts not of the forms $(2k+1)m$, $(2k+1)m\pm a$. Let $B_{k,a}(n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{\infty} f_i\cdot i$ (here f_i is the number of times the part i appears in the partition) with $f_1\leqq a-1$ and for all $i\geqq 1$,

$$f_i+f_{i+1}\leqq k-1.$$

Then $A_{k,a}(n)=B_{k,a}(n)$.

When $k=a=2$, (1.9) reduces to (1.1), and when $k=2$, $a=1$, (1.9) reduces to (1.2). Further theorems of this nature have been proved in subsequent papers [2], [3], [4].

The object of this paper is to generalize the Göllnitz-Gordon identities, (1.7) and (1.8), in the same manner that (1.9) generalizes (1.1) and (1.2). Our main result is the following theorem.

THEOREM 1. *Let a and k be integers with $0<a\leqq k$. Let $C_{k,a}(n)$ be the number of partitions of n into parts which are neither $\equiv 2 \pmod 4$ nor $\equiv 0, \pm(2a-1) \pmod{4k}$. Let $D_{k,a}(n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{\infty} f_i\cdot i$ with $f_1+f_2\leqq a-1$ and for all $i\geqq 1$,*

$$f_{2i-1}\leqq 1 \quad \textit{and} \quad f_{2i}+f_{2i+1}+f_{2i+2}\leqq k-1.$$

Then $C_{k,a}(n)=D_{k,a}(n)$.

When $k=a=2$, the theorem reduces to (1.7), and when $k=2$, $a=1$, the theorem reduces to (1.8). As an example, if $k=a=3$, the seven partitions enumerated by $D_{3,3}(8)$ are 8, $7+1$, $6+2$, $5+3$, $5+2+1$, $4+4$, $4+3+1$; the seven partitions enumerated by $C_{3,3}(8)$ are 8, $4+4$, $4+3+1$, $4+1+1+1+1$, $3+3+1+1$, $3+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1$.

In §2, we shall prove Theorem 1. In §3, we shall prove some analogues of the analytic form of the Rogers-Ramanujan identities [9, p. 290].

2. **Proof of Theorem 1.** We shall study the following functions. Throughout, $|q|<1$, and $x\neq -q^{-2n+1}$ for any $n\geqq 1$.

$$E_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2+(2k-2i+1)n} \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \right) \cdot \left\{ 1 - x^i q^{(2n+1)(2i-1)} \frac{(1+q^{2n+1})}{1+xq^{2n+1}} \right\}; \tag{2.1}$$

$$F_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2+(1-2i)n} \frac{(1-x^i q^{4in})}{(1-xq^{2n})} \cdot \prod_{j=1}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})}; \tag{2.2}$$

$$H_{k,i}(x) = F_{k,i}(x) \prod_{j=1}^{\infty} \frac{(1+xq^{2j-1})}{(1-xq^{2j})}; \tag{2.3}$$

$$J_{k,i}(x) = E_{k,i}(x) \prod_{j=1}^{\infty} \frac{(1+xq^{2j-1})}{(1-xq^{2j})}. \tag{2.4}$$

From the above definitions, we have immediately

$$F_{k,0}(x) = H_{k,0}(x) = 0, \tag{2.5}$$

and

$$H_{k,i}(0) = J_{k,i}(0) = 1, \qquad 1 \leqq i \leqq k. \tag{2.6}$$

To prove Theorem 1 we shall need the following lemmas.

LEMMA 1. $H_{k,i}(x) - H_{k,i-1}(x) = x^{i-1} J_{k,k-i+1}(x)$.

PROOF. We prove equivalently

$$F_{k,i}(x) - F_{k,i-1}(x) = x^{i-1} E_{k,k-i+1}(x) \cdot$$

$$F_{k,i}(x) - F_{k,i-1}(x)$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2+n} \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \right) (1-xq^{2n})^{-1}$$

$$\cdot (q^{-2in} - x^i q^{2in} - q^{-2in+2n} + x^{i-1} q^{2in-2n})$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2+n} \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \right) (1-xq^{2n})^{-1}$$

$$\cdot \left[\left\{ q^{-2in}(1-q^{2n}) \right\} + \left\{ x^{i-1} q^{2in-2n}(1-xq^{2n}) \right\} \right].$$

We now split our sum into two separate parts and replace n by $n+1$ in the first part. Hence

$$
\begin{aligned}
&F_{k,i}(x) - F_{k,i-1}(x) \\
&\quad = -\sum_{n=0}^{\infty} (-1)^n x^{kn+k} q^{2kn^2+4kn+2k+n+1-2in-2i} \frac{(1+q^{2n+1})}{(1+xq^{2n+1})} \\
&\qquad \cdot \prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \\
&\qquad + x^{i-1} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2-n+2in} \prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \\
&\quad = x^{i-1} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2+(2k-2(k-i+1)+1)n} \\
&\qquad \cdot \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \right) \\
&\qquad \cdot \left\{ 1 - x^{k-i+1} q^{(2n+1)(2(k-i+1)-1)} \frac{(1+q^{2n+1})}{1+xq^{2n+1}} \right\} \\
&\quad = x^{i-1} E_{k,k-i+1}(x).
\end{aligned}
$$

Thus we have Lemma 1.

LEMMA 2. $J_{k,i}(x) = H_{k,i}(xq^2) + xqH_{k,i-1}(xq^2)$.

PROOF. We prove equivalently

$$
E_{k,i}(x) = (1-xq^2)(1+xq)^{-1}(F_{k,i}(xq^2) + xqF_{k,i-1}(xq^2)).
$$

$$
\begin{aligned}
E_{k,i}(x) &= \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{2kn^2+(2k-2i+1)n} \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \right) \\
&\quad \cdot (1+xq^{2n+1})^{-1} \{ (1 - x^i q^{(2n+1)2i}) + xq^{2n+1}(1 - x^{i-1} q^{(2n+1)2(i-1)}) \} \\
&= \frac{(1-xq^2)}{(1+xq)} \sum_{n=0}^{\infty} (-1)^n (xq^2)^{kn} q^{2kn^2+(1-2i)n} \\
&\quad \cdot \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^2q^{2j+2})}{(1+xq^2q^{2j+1})(1-q^{2j+2})} \right) \\
&\quad \cdot (1-(xq^2)^i q^{4ni})(1-(xq^2)q^{2n})^{-1} \\
&\quad + \frac{xq(1-xq^2)}{(1+xq)} \sum_{n=0}^{\infty} (-1)^n (xq^2)^{kn} q^{2kn^2+(1-2(i-1))n} \\
&\quad \cdot \left(\prod_{j=0}^{n-1} \frac{(1+q^{2j+1})(1-xq^2q^{2j+2})}{(1+xq^2q^{2j+1})(1-q^{2j+2})} \right) \\
&\quad \cdot (1-(xq^2)^{i-1} q^{4n(i-1)})(1-(xq^2)q^{2n})^{-1} \\
&= (1-xq^2)(1+xq)^{-1}(F_{k,i}(xq^2) + xqF_{k,i-1}(xq^2)).
\end{aligned}
$$

Thus we have Lemma 2.

LEMMA 3. $J_{k,1}(x)=J_{k,k}(xq^2)$.

PROOF. In Lemma 1, put $i=1$; then (since $H_{k,0}(x)=0$ by (2.5)) $H_{k,1}(x)=J_{k,k}(x)$. In Lemma 2, put $i=1$; thus $J_{k,1}(x)=H_{k,1}(xq^2)$. Combining these two results, we obtain Lemma 3.

LEMMA 4. $J_{k,i}(x)-J_{k,i-1}(x)=x^{i-1}q^{2i-3}(qJ_{k,k-i+1}(xq^2)+J_{k,k-i+2}(xq^2))$.

PROOF. By Lemma 2,

$$\begin{aligned} J_{k,i}(x) - J_{k,i-1}(x) &= (H_{k,i}(xq^2) - H_{k,i-1}(xq^2)) \\ &\quad + xq(H_{k,i-1}(xq^2) - H_{k,i-2}(xq^2)) \\ &= x^{i-1}q^{2i-3}(qJ_{k,k-i+1}(xq^2) + J_{k,k-i+2}(xq^2)), \end{aligned}$$

where the second equation follows from Lemma 1. Thus we have Lemma 4.

We are now ready to treat our main theorem.

PROOF OF THEOREM 1. We may expand $J_{k,i}(x)$ as follows

$$J_{k,i}(x) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} c_{k,i}(M, N)x^M q^N, \tag{2.7}$$

where the double sum is subject to those conditions listed before (2.1).

Then by means of Lemmas 3 and 4 and equations (2.1) and (2.4), we easily verify that for $1 \leqq i \leqq k$

$$\begin{aligned} c_{k,i}(M, N) &= 1 \quad \text{if } M = N = 0, \\ &= 0 \quad \text{if either } M \leqq 0 \text{ or } N \leqq 0, \text{ and } M^2 + N^2 \neq 0, \end{aligned} \tag{2.8}$$

$$c_{k,1}(M, N) = c_{k,k}(M, N - 2M), \tag{2.9}$$

$$\begin{aligned} c_{k,i}(M, N) &- c_{k,i-1}(M, N) \\ &= c_{k,k-i+1}(M - i + 1, N - 2M) \\ &\quad + c_{k,k-i+2}(M - i + 1, N - 2M + 1), \qquad 1 < i \leqq k. \end{aligned} \tag{2.10}$$

One easily verifies by mathematical induction that the $c_{k,i}(M, N)$ for $1 \leqq i \leqq k$ are uniquely determined by (2.8), (2.9), and (2.10).

Let $p_{k,a}(M, N)$ denote the number of partitions of N into M parts of the form $N=\sum_{i=1}^{\infty} f_i \cdot i$ with $f_1+f_2 \leqq a-1$ and for all $i \geqq 1$, $f_{2i-1} \leqq 1$ and $f_{2i}+f_{2i+1}+f_{2i+2} \leqq k-1$. We wish to show that the $p_{k,i}(M,N)$ satisfy (2.8), (2.9), and (2.10). Now (2.8) is satisfied by definition.

As for (2.9), let us consider any partition enumerated by $p_{k,1}(M, N)$. Since neither 1 nor 2 appears, every summand is $\geqq 3$. Subtracting 2 from every summand, we obtain a partition of $N-2M$

into M parts with $f_1+f_2 \leqq k-1$ and $f_{2i}+f_{2i+1}+f_{2i+2} \leqq k-1$, $f_{2i-1} \leqq 1$. Thus we have a partition of the type enumerated by $p_{k,k}(M, N-2M)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,1}(M, N)$ and the partitions enumerated by $p_{k,k}(M, N-2M)$. Thus

$$p_{k,1}(M, N) = p_{k,k}(M, N-2M).$$

Finally we treat (2.10). We note that $p_{k,a}(M, N)-p_{k,a-1}(M, N)$ enumerates the number of partitions of N into M parts of the form $N=\sum_{i=1}^{\infty} f_i \cdot i$ with $f_1+f_2=a-1$ and $f_{2i}+f_{2i+1}+f_{2i+2} \leqq k-1$, $f_{2i-1} \leqq 1$. Hence either $f_2=a-1$, or $f_1=1$ and $f_2=a-2$. In case $f_2=a-1$, we see that $f_3+f_4 \leqq k-1-(a-1)$; subtracting 2 from every summand, we obtain a partition of $N-2M$ into $M-a+1$ parts with $f_1+f_2 \leqq (k-a+1)-1$ and for all $i \geqq 1$, $f_{2i-1} \leqq 1$ and $f_{2i}+f_{2i+1}+f_{2i+2} \leqq k-1$. Thus we have a partition of the type enumerated by $p_{k,k-a+1}(M-a+1, N-2M)$. In case $f_2=a-2$ and $f_1=1$, we see that $f_3+f_4 \leqq k-1-(a-2)$; subtracting 2 from every summand $\geqq 2$ and removing the summand 1, we obtain a partition of $N-(2M-1)$ into $M-a+1$ parts with $f_1+f_2 \leqq (k-a+2)-1$ and for all $i \geqq 1$, $f_{2i-1} \leqq 1$ and $f_{2i}+f_{2i+1}+f_{2i+2} \leqq k-1$. Thus we have a partition of the type enumerated by $p_{k,k-a+2}(M-a+1, N-2M+1)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,a}(M, N)-p_{k,a-1}(M, N)$ and the partitions enumerated by

$$p_{k,k-a+1}(M-a+1, N-2M) + p_{k,k-a+2}(M-a+1, N-2M+1).$$

Hence

$$\begin{aligned} p_{k,a}(M, N) - p_{k,a-1}(M, N) &= p_{k,k-a+1}(M-a+1, N-2M) \\ &\quad + p_{k,k-a+2}(M-a+1, N-2M+1). \end{aligned}$$

Thus by the comment following (2.10),

$$c_{k,i}(M, N) = p_{k,i}(M, N), \qquad 1 \leqq i \leqq k. \tag{2.11}$$

Thus for $1 \leqq a \leqq k$

$$\begin{aligned} 1 + \sum_{N=1}^{\infty} C_{k,a}(N)q^N &= \prod_{n=1;\, n \not\equiv 2 \pmod 4;\, n \not\equiv 0, \pm(2a-1) \pmod{4k}}^{\infty} (1-q^n)^{-1} \\ &= J_{k,a}(1) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} p_{k,a}(M, N)q^N \\ &= 1 + \sum_{N=1}^{\infty} D_{k,a}(N)q^N, \end{aligned}$$

where the second equation follows from Jacobi's identity [**9**, p. 283]. Therefore $C_{k,a}(N)=D_{k,a}(N)$. This concludes the proof of Theorem 1.

3. **Analytic identities.** Since $J_{1,1}(x)=J_{1,1}(xq^2)$ and $\lim_{h\to 0}J_{1,1}(h)=1$, we have

$$\begin{aligned} F_{1,1}(x) &= \sum_{n=0}^{\infty}(-1)^n x^n q^{2n^2-n}(1-xq^{4n})(1-xq^{2n})^{-1} \\ &\quad \cdot\prod_{j=0}^{n-1}\frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})} \\ &= E_{1,1}(x) = \prod_{n=1}^{\infty}\frac{(1-xq^{2n})}{(1+xq^{2n-1})}. \end{aligned} \tag{3.1}$$

It is easily deduced from Lemmas 3 and 4 that

$$J_{2,2}(x) = (1+xq)J_{2,2}(xq^2) + xq^2J_{2,2}(xq^4). \tag{3.2}$$

Expanding $J_{2,2}(x)$ in powers of x and using (3.2) and $J_{2,2}(0)=1$, we obtain

$$\begin{aligned} J_{2,2}(x) &= \sum_{m=0}^{\infty}\frac{x^m q^{m^2}(1+q)(1+q^3)\cdots(1+q^{2m-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2m})} \\ &= H_{2,1}(x) = \prod_{m=1}^{\infty}\frac{(1+xq^{2m-1})}{(1-xq^{2m})}\sum_{n=0}^{\infty}(-1)^n x^{2n}q^{4n^2-n} \\ &\quad \cdot\frac{(1-xq^{4n})}{(1-xq^{2n})}\prod_{j=0}^{n-1}\frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})}. \end{aligned} \tag{3.3}$$

(3.3) was the identity from which Göllnitz orignally deduced (1.7) and (1.8).

When $k=3$, the related q-identity becomes more complicated. If in equation (10.1) of [**5**, p. 431] we replace x by q and then put $a=x$, $f=-x$, we obtain

$$\begin{aligned} E_{3,3}(x) &= \prod_{m=1}^{\infty}(1-xq^m)\sum_{n=0}^{\infty}x^n q^{n^2}\prod_{j=0}^{n-1}\frac{(1+xq^{2j})}{(1-q^{j+1})(1-xq^{2j+1})(1+xq^j)} \\ &= F_{3,1}(x) = \sum_{n=0}^{\infty}(-1)^n x^{3n}q^{6n^2-n}\frac{(1-xq^{4n})}{(1-xq^{2n})} \\ &\quad \cdot\prod_{j=0}^{n-1}\frac{(1+q^{2j+1})(1-xq^{2j+2})}{(1+xq^{2j+1})(1-q^{2j+2})}. \end{aligned} \tag{3.4}$$

Putting $x=1$ in (3.1), we obtain a special case of Jacobi's identity [**9**, p. 283]. Putting $x=1$ in (3.3), we obtain equation (36) of [**11**, p.

155]. Putting $x=q^2$ in (3.3), we obtain equation (34) of [**11**, p. 155]. Putting $x=q^2$ in (3.4), we obtain equation (49) of [**11**, p. 156]. Putting $x=1$ in (3.4), we obtain equation (54) of [**11**, p. 157] (there appears to be a minor misprint in Slater's equation (54) which is easily corrected).

References

1. H. L. Alder, *The nonexistence of certain identities in the theory of partitions and compositions*, Bull. Amer. Math. Soc. **54** (1948), 712–722.

2. G. E. Andrews, *An analytic proof of the Rogers-Ramanujan-Gordon identities*, Amer. J. Math. **88** (1966), 844–846.

3. ———, *Partition theorems related to the Rogers-Ramanujan identities*, J. Combinatorial Theory (to appear).

4. ———, *Some new partition theorems*, J. Combinatorial Theory (to appear).

5. W. N. Bailey, *Some identities in combinatorial analysis*, Proc. London Math. Soc. (2) **49** (1947), 421–435.

6. H. Göllnitz, *Einfache Partitionen* (unpublished), Diplomarbeit W. S. 1960, Göttingen, 65 pp.

7. B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.

8. ———, *Some continued fractions of the Rogers-Ramanujan type*, Duke J. Math. **31** (1965), 741–748.

9. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, Oxford, 1960.

10. I. J. Schur, *Zur additiven Zahlentheorie*, S.-B. Akad. Wiss. Berlin (1926), 488–495.

11. L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **54** (1951–1952), 147–167.

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