A GENERALIZATION OF THE GÖLLNITZ-GORDON PARTITION THEOREMS

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1. Introduction. Among the most striking results in the theory of partitions are the Rogers-Ramanujan identities [9, p. 291]. These may be stated combinatorially as follows.

(1.1) The number of partitions of $n$ with minimal difference 2 is equal to the number of partitions of $n$ into parts of the forms $5m+1$ and $5m+4$.

(1.2) The number of partitions of $n$ into parts not less than 2, and with minimal difference 2, is equal to the number of partitions of $n$ into parts of the forms $5m+2$ and $5m+3$.

In 1926, I. J. Schur proved the following theorem which is similar to the above results [10].

(1.3) The number of partitions of $n$ of the form $n = b_1 + \cdots + b_s$, where $b_i - b_{i+1} \geq 3$, and $b_i - b_{i+1} > 3$ if $3 \mid b_i$, is equal to the number of partitions of $n$ into parts of the forms $6m+1$ and $6m+5$.

However, in 1948 H. L. Alder shut the door on further generalizations in this direction by proving the following three theorems [1]. Here $g_{d,m}(n)$ is the number of partitions of $n$ into parts differing by at least $d$, each part being greater than or equal to $m$.

(1.4) Let $S$ be any fixed set of positive integers, then $g_{d,m}(n)$ is not always equal to the number of partitions of $n$ into parts taken from $S$ if $d > 2$.

(1.5) Let $S$ be any fixed set of positive integers, then $g_{d,m}(n)$ is not equal to the number of partitions of $n$ into distinct parts taken from $S$ if $d > 1$.

(1.6) Let $S$ be any fixed set of positive integers, then the number of partitions of $n$ into parts differing by at least $d$ and where no consecutive multiples of $d$ appear is not equal to the number of partitions of $n$ into parts taken from $S$ if $d > 3$.

The case of (1.6) in which $d = 2$ was treated independently by H. Göllnitz [6, p. 33–34] in 1960 and by B. Gordon [8, p. 741] in 1965. They proved the following two identities.

(1.7) The number of partitions of any positive integer $n$ into parts $-1, 4, 7 \pmod{8}$ is equal to the number of partitions of the form $n = b_1 + \cdots + b_s$, where $b_i - b_{i+1} \geq 2$, and $b_i - b_{i+1} > 2$ if $b_i$ is even.

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(1.8) The number of partitions of any positive integer \( n \) into parts \( \equiv 3, 4, 5 \pmod{8} \) is equal to the number of partitions of the form \( n = b_1 + \cdots + b_s \) satisfying \( b_s \geq 3 \) in addition to the inequalities of (1.7).

A different form of generalization of the Rogers-Ramanujan identities was discovered in 1961 by B. Gordon [7]. He proved the following result.

(1.9) Let \( a \) and \( k \) be integers with \( 0 < a \leq k \). Let \( A_{k,a}(n) \) denote the number of partitions of \( n \) into parts not of the forms \((2k+1)m, (2k+1)m \pm a\). Let \( B_{k,a}(n) \) denote the number of partitions of \( n \) of the form \( n = \sum_{i=1}^{n} f_i \cdot i \) (here \( f_i \) is the number of times the part \( i \) appears in the partition) with \( f_1 \leq a - 1 \) and for all \( i \geq 1 \),

\[
f_i + f_{i+1} \leq k - 1.
\]

Then \( A_{k,a}(n) = B_{k,a}(n) \).

When \( k = a = 2 \), (1.9) reduces to (1.1), and when \( k = 2 \), \( a = 1 \), (1.9) reduces to (1.2). Further theorems of this nature have been proved in subsequent papers [2], [3], [4].

The object of this paper is to generalize the Göllnitz-Gordon identities, (1.7) and (1.8), in the same manner that (1.9) generalizes (1.1) and (1.2). Our main result is the following theorem.

**Theorem 1.** Let \( a \) and \( k \) be integers with \( 0 < a \leq k \). Let \( C_{k,a}(n) \) be the number of partitions of \( n \) into parts which are neither \( \equiv 2 \pmod{4} \) nor \( \equiv (2a-1) \pmod{4k} \). Let \( D_{k,a}(n) \) denote the number of partitions of \( n \) of the form \( n = \sum_{i=1}^{n} f_i \cdot i \) with \( f_1 + f_2 \leq a - 1 \) and for all \( i \geq 1 \),

\[
f_{2i-1} \leq 1 \quad \text{and} \quad f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1.
\]

Then \( C_{k,a}(n) = D_{k,a}(n) \).

When \( k = a = 2 \), the theorem reduces to (1.7), and when \( k = 2 \), \( a = 1 \), the theorem reduces to (1.8). As an example, if \( k = a = 3 \), the seven partitions enumerated by \( D_{3,3}(8) \) are 8, 7+1, 6+2, 5+3, 5+2+1, 4+4, 4+3+1; the seven partitions enumerated by \( C_{3,3}(8) \) are 8, 4+4, 4+3+1, 4+1+1+1+1+1, 3+3+1+1, 3+1+1+1+1+1+1, 1+1+1+1+1+1+1+1.

In \( \S2 \), we shall prove Theorem 1. In \( \S3 \), we shall prove some analogues of the analytic form of the Rogers-Ramanujan identities [9, p. 290].

2. **Proof of Theorem 1.** We shall study the following functions. Throughout, \( |q| < 1 \), and \( x \neq -q^{-2n+1} \) for any \( n \geq 1 \).
\[ E_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + (2k-2i+1)n} \left( \prod_{j=0}^{n-1} \frac{1 + q^{2j+1}}{1 + xq^{2j+1}} \right) \left( 1 + xq^{2j+2} \right) \]

\[ F_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + (1-2i)n} \left( \prod_{j=1}^{n-1} \frac{1 + q^{2j+1}}{1 + xq^{2j+1}} \right) \left( 1 + xq^{2j+2} \right) \]

\[ H_{k,i}(x) = F_{k,i}(x) \prod_{j=1}^{\infty} \frac{1 + xq^{2j-1}}{1 - xq^{2j}} \]

\[ J_{k,i}(x) = E_{k,i}(x) \prod_{j=1}^{\infty} \frac{1 + xq^{2j-1}}{1 - xq^{2j}} \]

From the above definitions, we have immediately

\[ F_{k,0}(x) = H_{k,0}(x) = 0, \]

and

\[ H_{k,i}(0) = J_{k,i}(0) = 1, \quad 1 \leq i \leq k. \]

To prove Theorem 1 we shall need the following lemmas.

**Lemma 1.** \( H_{k,i}(x) - H_{k,i-1}(x) = x^{-1} J_{k,k-i+1}(x) \).

**Proof.** We prove equivalently

\[ F_{k,i}(x) - F_{k,i-1}(x) = x^{-1} E_{k,k-i+1}(x). \]

\[ F_{k,i}(x) - F_{k,i-1}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + n} \left( \prod_{j=0}^{n-1} \frac{1 + q^{2j+1}}{1 + xq^{2j+1}} \right) \left( 1 + xq^{2j+2} \right) \left( 1 - xq^{2n} \right)^{-1} \]

\[ \cdot (q^{-2in} - x^i q^{2in} - q^{-2in+2n} + x^{-1} q^{2in-2n}) \]

\[ = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + n} \left( \prod_{j=0}^{n-1} \frac{1 + q^{2j+1}}{1 + xq^{2j+1}} \right) \left( 1 + xq^{2j+2} \right) \left( 1 - xq^{2n} \right)^{-1} \]

\[ \cdot \left[ \{ q^{-2in}(1 - q^{2n}) \} + \{ x^{-1} q^{2in-2n}(1 - xq^{2n}) \} \right]. \]

We now split our sum into two separate parts and replace \( n \) by \( n+1 \) in the first part. Hence
Thus we have Lemma 1.

**Lemma 2.** \( J_{k,i}(x) = H_{k,i}(xq^2) + xqH_{k,i-1}(xq^2). \)

**Proof.** We prove equivalently

\[
E_{k,i}(x) = (1 - xq^2)(1 + xq)^{-1}(F_{k,i}(xq^2) + xqF_{k,i-1}(xq^2)).
\]

\[
E_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^nx\kappa nq^{k n^2 + (2k-2i+1)n} \left( \prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})} \right) \left(1 - x^{k-i+1}q^{(2n+1)(2(k-i+1)-1)} \frac{(1 + q^{2n+1})}{1 + xq^{2n+1}} \right)
\]

\[
= x^{i-1}E_{k,k-i+1}(x).
\]

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Thus we have Lemma 2.

**Lemma 3.** \( J_{k,1}(x) = J_{k,k}(xq^2) \).

**Proof.** In Lemma 1, put \( i = 1 \); then (since \( H_{k,0}(x) = 0 \) by (2.5)) \( H_{k,1}(x) = J_{k,k}(x) \). In Lemma 2, put \( i = 1 \); thus \( J_{k,1}(x) = H_{k,1}(xq^2) \). Combining these two results, we obtain Lemma 3.

**Lemma 4.** \( J_{k,i}(x) - J_{k,i-1}(x) = x^{i-1}q^{2i-3}(qJ_{k,k-i+1}(xq^2) + J_{k,k-i+2}(xq^2)) \).

**Proof.** By Lemma 2,

\[
J_{k,i}(x) - J_{k,i-1}(x) = (H_{k,i}(xq^2) - H_{k,i-1}(xq^2))
+ xq(H_{k,i-1}(xq^2) - H_{k,i-2}(xq^2))
= x^{i-1}q^{2i-3}(qJ_{k,k-i+1}(xq^2) + J_{k,k-i+2}(xq^2)),
\]

where the second equation follows from Lemma 1. Thus we have Lemma 4.

We are now ready to treat our main theorem.

**Proof of Theorem 1.** We may expand \( J_{k,i}(x) \) as follows

\[
(2.7) \quad J_{k,i}(x) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} c_{k,i}(M, N)x^Mq^N,
\]

where the double sum is subject to those conditions listed before (2.1).

Then by means of Lemmas 3 and 4 and equations (2.1) and (2.4), we easily verify that for \( 1 \leq i \leq k \)

\[
(2.8) \quad c_{k,i}(M, N) = 1 \quad \text{if} \quad M = N = 0,
= 0 \quad \text{if either} \quad M \leq 0 \quad \text{or} \quad N \leq 0, \quad \text{and} \quad M^2 + N^2 \neq 0,
(2.9) \quad c_{k,1}(M, N) = c_{k,k}(M, N - 2M),
(2.10) \quad c_{k,i}(M, N) - c_{k,i-1}(M, N)
= c_{k,k-i+1}(M - i + 1, N - 2M)
+ c_{k,k-i+2}(M - i + 1, N - 2M + 1), \quad 1 < i \leq k.
\]

One easily verifies by mathematical induction that the \( c_{k,i}(M, N) \) for \( 1 \leq i \leq k \) are uniquely determined by (2.8), (2.9), and (2.10).

Let \( p_{k,a}(M, N) \) denote the number of partitions of \( N \) into \( M \) parts of the form \( N = \sum_{i=1}^{\infty} f_i \cdot i \) with \( f_1 + f_2 \leq a - 1 \) and for all \( i \geq 1, f_{2i-1} \leq 1 \) and \( f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1 \). We wish to show that the \( p_{k,i}(M,N) \) satisfy (2.8), (2.9), and (2.10). Now (2.8) is satisfied by definition.

As for (2.9), let us consider any partition enumerated by \( p_{k,1}(M, N) \). Since neither 1 nor 2 appears, every summand is \( \geq 3 \). Subtracting 2 from every summand, we obtain a partition of \( N - 2M \)
into $M$ parts with $f_1 + f_2 \leq k - 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$, $f_{2i-1} \leq 1$. Thus we have a partition of the type enumerated by $p_{k,k}(M, N - 2M)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,1}(M, N)$ and the partitions enumerated by $p_{k,k}(M, N - 2M)$. Thus

$$p_{k,1}(M, N) = p_{k,k}(M, N - 2M).$$

Finally we treat (2.10). We note that $p_{k,a}(M, N) - p_{k,a-1}(M, N)$ enumerates the number of partitions of $N$ into $M$ parts of the form $N = \sum_{i=1}^{N} f_i \cdot i$ with $f_1 + f_2 = a - 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$, $f_{2i-1} \leq 1$. Hence either $f_2 = a - 1$, or $f_1 = 1$ and $f_2 = a - 2$. In case $f_2 = a - 1$, we see that $f_3 + f_4 \leq k - 1 - (a - 1)$; subtracting 2 from every summand, we obtain a partition of $N - 2M$ into $M - a + 1$ parts with $f_1 + f_2 \leq (k - a + 1) - 1$ and for all $i \geq 1$, $f_{2i-1} \leq 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$. Thus we have a partition of the type enumerated by $p_{k,k-a+1}(M - a + 1, N - 2M)$. In case $f_2 = a - 2$ and $f_1 = 1$, we see that $f_3 + f_4 \leq k - 1 - (a - 2)$; subtracting 2 from every summand $\geq 2$ and removing the summand 1, we obtain a partition of $N - (2M - 1)$ into $M - a + 1$ parts with $f_1 + f_2 \leq (k - a + 2) - 1$ and for all $i \geq 1$, $f_{2i-1} \leq 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$. Thus we have a partition of the type enumerated by $p_{k,k-a+2}(M - a + 1, N - 2M + 1)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,a}(M, N) - p_{k,a-1}(M, N)$ and the partitions enumerated by

$$p_{k,k-a+1}(M - a + 1, N - 2M) + p_{k,k-a+2}(M - a + 1, N - 2M + 1).$$

Hence

$$p_{k,a}(M, N) - p_{k,a-1}(M, N) = p_{k,k-a+1}(M - a + 1, N - 2M)$$

$$+ p_{k,k-a+2}(M - a + 1, N - 2M + 1).$$

Thus by the comment following (2.10),

$$c_{k,i}(M, N) = p_{k,i}(M, N), \quad 1 \leq i \leq k.$$

Thus for $1 \leq a \leq k$

$$1 + \sum_{N=1}^{\infty} C_{k,a}(N) q^N = \prod_{n=1; n \equiv 2 (mod 4); n \equiv 0, \pm (2a-1) (mod 4k)} (1 - q^n)^{-1} \quad = J_{k,a}(1)$$

$$= \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} p_{k,a}(M, N) q^N$$

$$= 1 + \sum_{N=1}^{\infty} D_{k,a}(N) q^N,$$
where the second equation follows from Jacobi's identity [9, p. 283]. Therefore $C_{k,a}(N) = D_{k,a}(N)$. This concludes the proof of Theorem 1.

3. Analytic identities. Since $J_{1,1}(x) = J_{1,1}(xq^2)$ and $\lim_{a \to 0} J_{1,1}(a) = 1$, we have

$$F_{1,1}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2n^2-n}(1 - xq^{4n})(1 - xq^{2n})^{-1}$$

(3.1)

$$\prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})} = E_{1,1}(x) = \prod_{n=1}^{\infty} \frac{(1 - xq^{2n})}{(1 + xq^{2n-1})}.$$

It is easily deduced from Lemmas 3 and 4 that

(3.2) $J_{2,2}(x) = (1 + xq)J_{2,2}(xq^2) + xq^2J_{2,2}(xq^4)$.

Expanding $J_{2,2}(x)$ in powers of $x$ and using (3.2) and $J_{2,2}(0) = 1$, we obtain

$$J_{2,2}(x) = \sum_{n=0}^{\infty} \frac{x^n q^{n^2}(1 + q)(1 + q^3) \cdots (1 + q^{2m-1})}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2m})}$$

(3.3)

$$H_{2,1}(x) = \prod_{m=1}^{\infty} \frac{(1 + xq^{2m-1})}{(1 - xq^{2m})} \sum_{n=0}^{\infty} (-1)^n x^n q^{4n^2-n}$$

$$\cdot \frac{(1 - xq^{4n})}{(1 - xq^{2n})} \prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})}.$$

(3.3) was the identity from which Göllnitz originally deduced (1.7) and (1.8).

When $k = 3$, the related $q$-identity becomes more complicated. If in equation (10.1) of [5, p. 431] we replace $x$ by $q$ and then put $a = x$, $f = -x$, we obtain

$$E_{3,3}(x) = \prod_{m=1}^{\infty} (1 - xq^m) \sum_{n=0}^{\infty} x^n q^{n^2} \prod_{j=0}^{n-1} \frac{(1 + xq^{2j})}{(1 - q^{j+1})(1 - xq^{2j+1})(1 + xq^j)}$$

(3.4)

$$F_{3,1}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{6n^2-n} \frac{(1 - xq^{4n})}{(1 - xq^{2n})} \cdot \prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})}.$$

Putting $x = 1$ in (3.1), we obtain a special case of Jacobi's identity [9, p. 283]. Putting $x = 1$ in (3.3), we obtain equation (36) of [11, p.
Putting $x = q^2$ in (3.3), we obtain equation (34) of [11, p. 155]. Putting $x = q^2$ in (3.4), we obtain equation (49) of [11, p. 156]. Putting $x = 1$ in (3.4), we obtain equation (54) of [11, p. 157] (there appears to be a minor misprint in Slater's equation (54) which is easily corrected).

REFERENCES


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