THE SHARPENING OF A RESULT CONCERNING
PRIMITIVE IDEALS OF AN ASSOCIATIVE RING

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The importance of the concept of primitive ideals of associative rings consists in the well-known theorem stating that every semisimple ring $A$ is a subdirect sum of primitive rings $B_v$, where a ring $A$ is called semisimple (in the sense of Jacobson) if the Jacobson radical, i.e. the intersection of all primitive ideals, coincides with the zero ideal $(0)$, and a ring $B_v$ is called primitive if the ideal $(0)$ is a primitive ideal of $B_v$. (Cf. N. Jacobson, Structure of rings, Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.)

Some new characterizations were recently given for the Jacobson radical of a ring $A$. For instance, A. Kertész [3] has shown in these Proceedings (generalizing an observation of L. Fuchs [1]) that the Jacobson radical $J$ of a ring $A$ consists of exactly those elements $x$ of $A$ for which the product $yx$ lies with every $y \in A$ in the Frattini $A$-submodule of the ring $A$, as of an $A$-right module $A$ for itself (cf. also Hille [2]). Furthermore A. Kertész [4] has shown that $J$ is the intersection of all those maximal right ideals $R$ of $A$ for which there must exist, for any element $x \in R$ ($x \in A$), a second element $y \in A$ with $yx \in R$; that is, those right ideals for which $A^{-1}R \subseteq R$ holds, where $X^{-1}R = \{y; y \in A, Xy \subseteq R\}$ for an arbitrary subset $X$ of $A$. Furthermore, let $L \cdot Y^{-1}$ denote the subset $\{z; z \in A, zY \subseteq L\}$.

Every modular right ideal $R$ of $A$ is quasi-modular in the sense that $A^{-1}R \subseteq R$ holds. The concept of quasi-modularity of right ideals $R$ was introduced in [6]. Solving a problem proposed by Kertész [4] I have shown in [6] the existence of an associative ring which has a quasi-modular maximal but not a modular right ideal. In my other paper [7] a two-sided ideal $Q$ of $A$ is called quasi-primitive if there exists a quasi-modal maximal right ideal $R$ of $A$ with $Q = A^{-1}R \subseteq R$. Obviously every primitive ideal is also quasi-modal in $A$, and almost trivially every artin ring with $(0)$ quasi-primitive ideal is a total matrix ring over a skew field. Furthermore, any quasi-primitive ideal is clearly a prime ideal, and any commutative ring with $(0)$ quasi-primitive ideal is a field.

Solving a problem of my colleague Dr. Steinfeld, I have proved in [7] that the Jacobson radical $J$ of $A$ must coincide with the intersec-

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tion of all quasi-primitive ideals. There are two proofs of this fact in [7], an entirely elementary proof without quasi-regular element and irreducible modules, and (in a footnote) a second short proof with quasi-regular elements too.

In my note [7] some open problems on quasi-primitive ideals are mentioned, which have recently been solved completely by Dr. Steinfeld. He has shown that the concepts of primitivity and quasi-primitivity of ideals of arbitrary associative rings must coincide. Using a lemma which is proved but not explicitly announced in [7], Dr. Steinfeld has proved that there exists for every fixed quasi-modular maximal right ideal $R$ of $A$ an element $X$ of $A$ for which the right ideal quotient $R_X = \{x\} \cdot R$ is a modular maximal right ideal of $A$ such that $A^{-1}R = A^{-1}R_X = (xA)^{-1}R$, which means that every quasi-primitive ideal $Q = A^{-1}R$ is by $Q = A^{-1}R_X$ also primitive in $A$.

This result of Dr. Steinfeld can be sharpened as follows:

**Theorem.** If $R$ is a quasi-modular maximal right ideal of an arbitrary associative ring, and if $x \in A$ is an arbitrary element of $A$ with the condition $x \in R$, then the quasi-primitive ideal $Q = A^{-1}R$ coincides with the primitive ideal $P_x = A^{-1}R_x = (xA)^{-1}R$ of $A$ (instead of a single $x$ for any $x \in R$).

**Proof.** In my note [7] it is shown that $R_x = \{x\} \cdot R$ is a modular maximal right ideal of $A$ for every quasi-modular maximal right ideal $R$ of $A$ and for every $x \in A$ with $x \in R$. Namely, $R_x = \{x\} \cdot R$ is a right ideal of $A$. By the quasi-modularity of $R$, $A^2 + R = A$, and therefore we obtain $RA^{-1} = R$; that is, $xA + R = A$ for any $x \in R$, $x \in A$. Since there exists for $x \in R$ an element $y \in A$ with $xy \in R$, the right ideal $R_x$ has the property $y \in R_x$, i.e. $R_x \neq A$. If $z \in A$ is any element with $z \in R_x$, one has by $xz \in R$ obviously $xzA + R = A$, and thus for any $b \in A$ the existence of $a \in A$ and $r \in R$ with $xza + r = xb$, and thereby also $x(b - za) = r \in R$, $b - za \in R_x$, $b \in zA + R_x$ and $A = zA + R_x$, which means the maximality of $R_x$ in $A$. Moreover, one has $xza_1 + r_1 = x$ with some $a_1 \in A$ and $r_1 \in R$, which implies $x(1 - za_1)A \subseteq R$, consequently $(1 - za_1)A \subseteq R_x$ and the modularity of the maximal right ideal $R_x$ of $A$.

By $xA + R = A$ and $A((xA)^{-1}R) = (xA + R)((xA)^{-1}R) \subseteq R$ one has on one side $(xA)^{-1}R \subseteq A^{-1}R$. On the other hand the condition $y \in A^{-1}R$ implies by $A^{-1}R = (xA + R)^{-1}R$ obviously $xAy \subseteq R$, that is $y \in (xA)^{-1}R$, and thus holds $A^{-1}R = (xA)^{-1}R$ for every $x \in R$ ($x \in A$). But one has almost trivially $(xA)^{-1}R = A^{-1}\{x\}^{-1}R = A^{-1}R_x$ too, which means that $Q = A^{-1}R = (xA)^{-1}R$ and $P_x = A^{-1}R_x$ must be for every $x \in R$ the same primitive ideals of $A$. Q.E.D.
References


