VARIOUS TYPES OF NORM-DETERMINING MANIFOLDS

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1. **Introduction.** The objectives of this paper are to prove some theorems, reminiscent of those obtained by Dixmier [3] and Ruston [4], concerning some norm-determining manifolds, and to introduce a family of Banach spaces, called spaces with property D, and show that this property is a generalization of some well-known conditions; for example, reflexiveness. Absolutely total hyperplanes are characterized as those whose generating functional is orthogonal to the space's canonical image, and d-manifolds are characterized in terms of the types of norms definable on them. Some special nonreflexive spaces, in particular l_1, are shown to have property D. Any subsequent reference to Dixmier or Ruston will refer to [3] or [4] respectively.

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2. **Preliminary definitions.** The term "linear space" refers to a nondegenerate linear space over the field of real numbers. If S is a linear space, then S^+ denotes the set of all linear functionals on S, and $N(S)$ denotes the origin in S; if S is normed, then $U(S)$ denotes the set of points in S with norm 1 and J denotes the canonical map of S into its second conjugate space, S^{**}.

Suppose that S is a linear space, that K is a linear manifold in S^+, and that h is a norm on K. Then if x is in S, $A(x, K, h) = \{|f(x)| \mid f$ is in K and $h(f) \leqq 1\}$. If for each x in S, $A(x, K, h)$ is bounded, then $g(\cdot, K, h)$ denotes the transformation from S into E_1 such that if x is in S, then $g(x, K, h) = \text{l.u.b. } A(x, K, h)$. If S is normed, K is contained in S^* and h is the usual norm on S^*, then $A(x, K, h)$ is denoted by $A(x, K)$ and $g(\cdot, K, h)$ is denoted by $g(\cdot, K)$.

3. **Some properties of absolutely total manifolds.** In this section we suppose that S is a normed linear space.

DEFINITION 3.1. Suppose that K is a linear manifold in S^*. The statement that K is absolutely total means that if x is in S, then $g(x, K) = \|x\|$. (An absolutely total manifold is one which is of characteristic 1 in the sense of Dixmier.)

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LEMMA 3.1. *Suppose that K is a dense linear manifold in S^*. Then K is absolutely total.*

PROOF. Suppose that x is in S. There is a point f of S^* such that $|f| \leqq 1$ and $|f(x)| = \|x\|$. Since K is dense in S^*, there is a sequence $f_1, f_2, \cdots$ each term of which is a point of K with norm $\leqq 1$ and which converges to f in S^*. Then $|f_1(x)|, |f_2(x)|, \cdots$ converges to $|f(x)| = \|x\|$, and $|f_n(x)|$ belongs to $A(x, K)$ for each n, so $\|x\|$ is a point or a limit point of $A(x, K)$. Since $\|x\|$ is an upper bound of $A(x, K)$, $\|x\| = \text{l.u.b. } A(x, K) = g(x, K)$. Thus K is absolutely total.

Ruston proved that if S is complete, it is reflexive if and only if S^* contains no proper closed total linear manifold. Using this with Lemma 3.1 we get a more complete theorem.

THEOREM 3.1. *If S is complete, the following two statements are equivalent*:

(1) *S is reflexive*

(2) *if K is a linear manifold in S^*, then the following three statements are equivalent*:

(a) *K is total,*

(b) *K is absolutely total,*

(c) *K is dense in S^*.*

PROOF. Suppose that S is reflexive, and K is a linear manifold in S^*. If K is total, $\overline{K}$ is total and closed, so $\overline{K} = S^*$ and K is dense in S^*. If K is dense in S^* it is absolutely total by Lemma 3.1. Clearly, any absolutely total manifold is total. Thus statement (2) is true. Conversely, if statement (2) is true, S^* contains no proper closed total manifold and S is reflexive.

Part (a) of statement (2) cannot be omitted in Theorem 3.1 since Ruston has shown that each absolutely total manifold in c_0^* is dense in c_0^*. However, it can be omitted in the case of second conjugate spaces.

THEOREM 3.2. *If S is complete, the following two statements are equivalent*:

(1) *S is reflexive;*

(2) *if K is an absolutely total linear manifold in S^{**}, then K is dense in S^{**}.*

PROOF. Suppose that statement (2) is true. Dixmier showed that $J(S)$ is absolutely total in S^{**}, so it is dense in S^{**}. Since S is complete, $J(S)$ is closed in S^{**}. Hence $J(S) = S^{**}$ and S is reflexive. Conversely, if S is reflexive, S^* is reflexive and each absolutely total manifold in S^{**} is dense in S^{**} by Theorem 3.1.

The final theorem of this section characterizes the points of S^{**} whose zero hyperplane is an absolutely total manifold in S^*. Since the next lemma is given as an exercise in [**5**, p. 105], it is stated here without proof.

LEMMA 3.2. *If K is a linear manifold in S^* and x is in S, then $g(x, K)=d(J_x, K^0)$, where K^0 is the set of points in S^{**} which are zero on K.*

DEFINITION 3.2. Suppose that each of x and y is in S and that V is a subset of S. The statement that y is orthogonal to x means that if k is a number, then $\|x\| \leqq \|x-ky\|$. The statement that y is orthogonal to V means that y is orthogonal to each point of V.

THEOREM 3.3. *Suppose that S is complete, that F is a point of S^{**} not in $J(S)$, and that K is the zero hyperplane of F. Then the following two statements are equivalent*:

(1) *K is absolutely total*;

(2) *F is orthogonal to $J(S)$.*

PROOF. Suppose that K is absolutely total. Suppose moreover that J_x is a point of $J(S)$ and that k is a number. Since $\|x\| = g(x, K)$, $|J_x| = d(J_x, K^0)$ by Lemma 3.2. But kF is in K^0, so $|J_x| \leqq |J_x - kF|$, and F is orthogonal to J_x. Hence F is orthogonal to $J(S)$. Conversely, suppose that F is orthogonal to $J(S)$ but that K is not absolutely total. Then there is a point x of S such that $g(x, K) < \|x\|$. Then $d(J_x, K^0)$ is less than $|J_x|$, and there is a point G of K^0 such that $|J_x - G| < |J_x|$. But K is a hyperplane and each of F and G is zero on K, so G is a multiple of F, say $G=kF$. Then $|J_x - kF| < |J_x|$ and F is not orthogonal to J_x, resulting in a contradiction. The statements are thus equivalent.

4. Some properties of d-manifolds.

In this section S denotes a linear space.

DEFINITION 4.1. If S is normed, and K is a linear manifold in S^*, the statement that K is a d-manifold means that if x is in S, there is a point f of K such that $|f| = 1$ and $f(x) = \|x\|$.

Any d-manifold is absolutely total. To obtain an absolutely total manifold which is not a d-manifold, consider the set $B = \{b_1, b_2, \cdots\}$ of points in H^* (H=real Hilbert space) where $b_n(x) = x_n$ for each point $x = (x_1, x_2, \cdots)$ of H, and let K denote the linear span of B. K is absolutely total since it is dense in H^* but there is no point of K with norm one which maps the point x of H such that $x_n = 1/(2^n)^{1/2}$ onto its norm.

We now give a general characterization of d-manifolds.

THEOREM 4.1. *If K is a total linear manifold in S^+, then the following two statements are equivalent*:

(1) *there exists a norm g on S such that K is a d-manifold in $(S, g)^*$;*

(2) *there exists a norm h on K such that if x is in S, then $A(x, K, h)$ is closed and bounded.*

PROOF. If statement (1) is true, we may take h to be the usual norm in $(S, g)^*$ to obtain statement (2). Suppose that statement (2) is true. Let g denote the transformation $g(\cdot, K, h)$. Since K is total, $g(x)>0$ if x is not $N(S)$. Clearly $g(N(S))=0$. Using the definition of g, it is a straightforward argument to show that g is a seminorm on S. Hence g is a norm on S. If f is a point of K different from $N(S^+)$, x is in S, and $f'=[1/h(f)]f$, then

$$|f(x)| = h(f)|f'(x)| \leqq h(f)g(x).$$

Thus f is bounded relative to g and $|f| \leqq h(f)$. Hence K is contained in $(S, g)^*$. Suppose x is in S. Since $A(x, K, h)$ is closed, it contains its least upper bound, $g(x)$. Then there is a point f of K such that $h(f)\leqq 1$ and $|f(x)|=g(x)$. Then $|f|\leqq 1$ and $|f(x)|=g(x)$. Hence K is a d-manifold in $(S, g)^*$.

5. **Property** D. In this section, S denotes a nondegenerate Banach space.

DEFINITION 5.1. The statement that S has property D means that if K is a d-manifold in S^*, then K is dense in S^*.

DEFINITION 5.2. If x is in S, the statement that x is a smooth point means that there is only one point f of S^* (called the unique extension of x) such that $|f|=1$ and $f(x)=\|x\|$. The statement that S is smooth means that each point of $U(S)$ is a smooth point.

THEOREM 5.1. *Suppose that S has a smooth point and that E denotes the set of unique extensions of smooth points in $U(S)$. If the linear span of E (denoted by* L(E)*) is dense in S^*, then S has property* D.

PROOF. Suppose that K is a d-manifold in S^*. Let f be a point of E. Then f is the unique extension of a smooth point x. Since K is a d-manifold, there is a point g of K with norm one such that $g(x)=\|x\|$. Then g is f since x is a smooth point. Then E is contained in K, so L(E) is contained in K and K is dense in S^*. Hence S has property D.

COROLLARY 5.1. *If S is smooth, then S has property* D.

PROOF. The set E of Theorem 5.1 contains all the points of $U(S^*)$ which attain their norm, so L(E) contains all the points of S^* which attain their norm. Bishop and Phelps proved [1] that this set is dense in S^*, so L(E) is dense in S^* and S has property D.

COROLLARY 5.2. *If S^* is strictly convex, then S has property* D.

Since each absolutely total manifold in c_0^* is dense in c_0^*, it follows that c_0 has property D. Also, it follows from Theorem 3.1 that any reflexive space has property D. In particular then, l_p for $1<p<\infty$ has property D. We will now show that l_1 does also. (I do not know whether l_∞ has property D.)

DEFINITION 5.3. Suppose that $V=\{v_1, v_2, \cdots, v_k\}$ is a finite number set with only k elements. The statement that the number w is a proper sum of V means that there is a finite sequence $q_1, q_2, \cdots, q_k$ with only k terms such that each q_i is either 1 or 2 and such that $w=\sum_{i=1}^{k}[(v_i)(-1)^{q_i}]$.

The proof of the next lemma is a straightforward induction argument and is omitted here.

LEMMA 5.1. *Suppose that n is a positive integer such that $n-2$ is a multiple of* 4. *Then if m is an even positive integer less than or equal to $n-2$, m is a proper sum of the set of integers in the interval* $[1, n-2]$.

LEMMA 5.2. *Suppose that V denotes the set to which the point $f=(f_1, f_2, \cdots)$ of l_∞ belongs if and only if $|f_n|=1$ for each n. Then* $\mathrm{L}(V)$ *is dense in* l_∞.

PROOF. Suppose that $x=(x_1, x_2, \cdots)$ is a point of l_∞. Let $[a, b]$ denote a number interval containing the range of x. Suppose that h is a positive number. Let n be a positive integer such that $n-2$ is a multiple of 4 and such that the set $\{a, a+2(b-a)/n, a+4(b-a)/n, \cdots, b\}$ partitions $[a, b]$ into subintervals of length less than $h/2$. Then for each term x_i of x, there is an even positive integer $m(x_i)\leqq n-2$ such that $|x_i-[a+m(x_i)(b-a)/n]|<h/2$. By Lemma 5.1, $m(x_i)$ is a proper sum of $1, 2, \cdots, n-2$, say $m(x_i) = \sum_{k=1}^{n-2}[(k)(-1)^{q_k(x_i)}]$. Let F_0 denote the point of l_∞ each of whose terms is 1 and for $j=1, 2, \cdots, n-2$ let F_j denote the point of l_∞ whose kth term is $(-1)^{(jqxk)}$. Then the point

$$F = aF_0 + \sum_{j=1}^{n-2} [(j(b-a)/n)F_j]$$

is in $\mathrm{L}(V)$ and $\|x-F\|<h$. Thus $\mathrm{L}(V)$ is dense in l_∞.

THEOREM 5.2. *The space l_1 has property* D.

PROOF. Identify the points of l_1^* with the points of l_∞ in the usual manner. Then each point $f=(f_1, f_2, \cdots)$ of the set V in Lemma 5.2 is the unique extension of the smooth point $x=(x_1, x_2, \cdots)$ of l_1 such that $x_i=1/2^i$ if $f_i=1$ and $x_i=-1/2^i$ if $f_i=-1$. Since $\mathrm{L}(V)$ is dense in l_∞ it follows from Theorem 5.1 that l_1 has property D.

By arguments similar to the preceding ones, the space c of convergent number sequences can be shown to have property D. The space c^* has property D since it is congruent with l_1.

While obtaining the preceding theorems, I had conjectured that every Banach space had property D. That this is not the case was pointed out to me by Professor W. P. Coleman, who proved that the space C of continuous real valued functions from $[0, 1]$ failed to have property D when given the norm $\|P\| = \max_{t \in [0,1]} |P(t)|$. The construction of his counterexample may be outlined as follows: for each number x in $[0, 1]$ let f_x denote the point of C^* such that $f_x(P) = P(x)$ for each P in C; let $R = \{f_x | x \text{ in } [0, 1]\}$ and let $K = \mathrm{L}(R)$. Then K is a d-manifold in C^*, but the point g of C^* such that $g(P) = \int_0^1 P dI$ for each P in C is not in the closure of K.

M. M. Day proved [2] that every separable Banach space is isomorphic to a space which is strictly convex and smooth. Then every separable Banach space (and hence C) is isomorphic to a space with property D, and property D is not preserved under isomorphisms. It would be nice to know whether every Banach space is isomorphic to a space with property D.

References

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