ON STRONG RIEZ$^\text{S}$ SUMMABILITY FACTORS OF INFINITE SERIES. I

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(1.1) Let $\sum_{n=1}^{\infty} a_n$ be a given infinite series, and $\{\lambda_n\}$ an increasing sequence of positive numbers, tending to infinity with $n$. We write

\[
A^0(\omega) = A(\omega) = \sum_{\lambda_n < \omega} a_n,
\]

\[
A^k(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^k a_n = \int_0^\omega (\omega - t)^k A(t),
\]

\[
A^k(\omega) = 0 \quad \text{for } \omega \leq 1, \text{ and } k > -1.
\]

The series $\sum a_n$ is said to be summable $(R, \lambda, k)$ to the sum $s$, if $\lim_{\omega \to \infty} \omega^{-k} A^k(\omega) = s$. The given series is said to be strongly summable $(R, \lambda, k)$, or simply summable $[R, \lambda, k]$ to the sum $s$, if

\[
\int_0^\omega |x^{-k+1} A^{k-1}(x) - s| \, dx = o(\omega);
\]

where $k > 0$. The series $\sum a_n$ is said to be strongly summable $(R, \lambda, k)$ with index $m > 0$, or summable $[R, \lambda, k, m]$ to the sum $s$, if

\[
\int_0^\omega |x^{-k+1} A^{k-1}(x) - s|^m \, dx = o(\omega),
\]

where $k > 0$ and $km' > 1$, $(1/m + 1/m' = 1)$.

(1.2) The classical second theorem of consistency due to Hardy and Riesz [3] is to the effect that if $\sum a_n$ is summable $(R, \lambda, k)$ and $\lambda_n = e^{\mu n}$, then it is also summable $(R, \mu, k)$ to the same sum. Later Hardy [2] generalized this theorem and proved:

THEOREM A. If the series $\sum a_n$ is summable $(R, \lambda, k)$ and $\mu$ is a logarithmico-exponential function (briefly an $L$-function) of $\lambda$, tending to infinity with $\lambda$, such that $\mu = O(\lambda^\Delta)$, where $\Delta$ is some constant, then $\sum a_n$ is summable $(R, \mu, k)$.

Finally Hirst [4] removed the limitation on $\mu$ of being an $L$-function, replacing $\mu$ by a more general function $\phi(t)$, and proved:

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Theorem B. If $\sum a_n$ is summable $(R, \lambda, k)$, then it is summable $(R, \mu, k)$ to the same sum, where $\mu = \phi(\lambda)$, and $\phi(t)$ is a function which increases steadily to infinity with $t$ and is a $(k+1)$th indefinite integral for $t>0$, such that

$$\int_0^z t^n |\phi^{(n+1)}(t)| \, dt = O\left\{\phi(x)\right\}; \quad n = 1, 2, \ldots, k.$$ 

The following theorem on strong Riesz summability factors was proved by Borwein and Shawyer [1].

Theorem C. For all $k \geq 1$, if

(i) $\phi(t)$ is an L-function,

(ii) $1/\omega = O\left\{\phi'(\omega)/\phi(\omega)\right\}$,

(iii) $\psi(\omega) = \left\{\phi(\omega)/\omega\phi'(\omega)\right\}^k$,

then $\sum a_n \psi(\lambda_n)$ is summable $[R, \phi(\lambda), k]$ whenever $\sum a_n$ is summable $[R, \lambda, k]$.

The object of this paper is to establish a more general summability factor theorem for strong Riesz summability, which includes as a particular case Theorem C.

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(2.1) In the following we take functions $\phi(t)$ and $\psi(t)$ to be defined in $(0, \infty)$ and to be as many times differentiable as required. In addition let $\phi(t)$ be nonnegative, monotone increasing, and tending to infinity with $t$.

We establish the following theorem.

Theorem. Let $k$ be a positive integer, and $\phi(t)$ and $\psi(t)$ be $(k+1)$th indefinite integrals for $t>0$. If there is a positive, nondecreasing function $\gamma(t)$ such that

(i) $\gamma(t) = O(t)$ in $(a, \infty)$; $a > 0$,

(ii) $t^n \psi^{(n)}(t) = O\left\{\gamma(t)\right\}^{k-n}; \quad n = 0, 1, \ldots, k; \quad t > a$,

(iii) $\{\gamma(t)\}^{r\phi^{(n)}(t)} = O\left\{\phi(t)\right\}; \quad n = 1, 2, \ldots, k; \quad t > a$,

then $\sum a_n \psi(\lambda_n)$ is summable $[R, \phi(\lambda), k, m]$, whenever $\sum a_n$ is summable $[R, \lambda, k, m]$, where $m \geq 1$ and $km' > 1$.

(2.2) The following lemmas will be required in the proof of our theorem.

Lemma 1 [3]. If $k$ is a positive integer, then $A(t) = (1/k!)(d/dt)^k A^k(t)$.

Lemma 2 [5]. If $\sum a_n$ is summable $[R, \lambda, k, m]$ for $m \geq 1$, then it is summable $(R, \lambda, k)$. 

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Lemma 3 [8]. If \( n \) is a positive integer and \( m \neq 0 \), then the \( n \)th derivative of \( \{f(x)\}_m \) is a sum of the constant multiples of a finite number of terms of the form
\[
\{f(x)\}^{m-q} \prod_{p=1}^{n} \{f^{(p)}(x)\}^{\alpha_p},
\]
where \( 1 \leq q \leq n \) and the \( \alpha \)'s are nonnegative integers such that
\[
\sum_{p=1}^{n} \alpha_p = q \quad \text{and} \quad \sum_{p=1}^{n} p\alpha_p = n.
\]
If \( m \) is a positive integer, then \( 1 \leq q \leq \min(m, n) \).

3. Proof of the theorem. We may suppose without loss of generality that the sum of the given series is zero. Then the hypothesis of summability \( [R, \lambda, k, m] \) of \( \sum a_n \) reduces to
\[
(3.1) \quad \int_{0}^{\omega} |A^{k-1}(t)|^m dt = o[\omega^{(k-1)m+1}].
\]
And for summability \( [R, \phi(\lambda), k, m] \) of \( \sum a_n \psi(\lambda_n) \) we must show
\[
(3.2) \quad \int_{0}^{\omega} \phi'(t) |B^{k-1}\{\phi(t)\}|^m dt = o[\phi(\omega)^{(k-1)m+1}],
\]
where
\[
(3.3) \quad B^{k-1}\{\phi(t)\} = \int_{0}^{t} \{\phi(u) - \psi(u)\}^{k-1}\psi(u) dA(u).
\]
Integrating by parts, we have
\[
B^{k-1}\{\phi(t)\} = - \int_{0}^{t} (\partial/\partial u)[\{\phi(t) - \phi(u)\}^{k-1}\psi(u)] A(u) du.
\]
And by Lemma 1,
\[
B^{k-1}\{\phi(t)\}
(3.4) \quad = - \frac{1}{(k-1)!} \int_{0}^{t} \frac{\partial}{\partial u} [\{\phi(t) - \phi(u)\}^{k-1}\psi(u)] \left( \frac{d}{du} \right)^{k-1} A^{k-1}(u) du.
\]
Since \( A^k(u) \) and its first \( (k-1) \) derivatives vanish at \( u = 0 \), integrating (3.4) by parts \( (k-1) \) times, we get
\[
B^{k-1}\{\phi(t)\} = A^{k-1}(t) \psi(t) \{\phi'(t)\}^{k-1}
(3.5) \quad + \frac{(-1)^k}{(k-1)!} \int_{0}^{t} A^{k-1}(u) \left( \frac{\partial}{\partial u} \right)^k [\{\phi(t) - \phi(u)\}^{k-1}\psi(u)] du.
\]
Thus to prove the theorem, it is sufficient to prove that

\[(3.6) \quad \int_0^\omega |A^{k-1}(t)|^m \{\psi(t)\}^m \{\phi'(t)\}^{(k-1)m+1} dt = o\left[\phi(\omega)\right]^{(k-1)m+1}
\]

and

\[(3.7) \quad \int_0^\omega \phi'(t) \left| \int_0^t A^{k-1}(u) \left(\frac{\partial}{\partial u}\right)^k \left[\phi(t) - \phi(u)\right]^{k-1} \psi(u) du \right|^m dt = o\left[\phi(\omega)\right]^{(k-1)m+1}.
\]

**Proof of (3.6).** Let

\[J = \int_0^\omega |A^{k-1}(t)|^m \{\psi(t)\}^m \{\phi'(t)\}^{(k-1)m+1} dt
\]

By hypotheses (ii) and (iii) of the theorem and the fact that $A^k(\omega) = 0$ for $\omega \leq 1$ and $k > -1$, we have

\[J = O\left(\int_1^\omega |A^{k-1}(t)|^m \left(\frac{\gamma(t)}{t}\right)^{k-m} \phi'(t) \gamma(t) dt\right)
\]

\[= O\left(\left[\phi(\omega)\right]^{(k-1)m+1} \int_1^\omega |A^{k-1}(t)|^m t^{-k+m} \gamma(t) \gamma(t)^{m-1} dt\right)
\]

since $k \geq 1$, $m \geq 1$ and $\gamma(t)$ is a nondecreasing positive function.

Integrating the last expression by parts and using (3.1) we have:

\[J = o\left[\phi(\omega)\right]^{(k-1)m+1} \gamma(\omega) \gamma(\omega)^{m-1} \omega^{1-m}
\]

by (i) of the theorem.

**Proof of (3.7).** Note that $(\partial/\partial u)^k \left[\phi(t) - \phi(u)\right]^{k-1} \psi(u)$ is a sum of constant multiples of terms of type

\[\psi^{(\nu)}(u) \left(\frac{\partial}{\partial u}\right)^{k-\nu} \phi(t) - \phi(u) \right|^{k-1}; \quad 0 \leq \nu \leq k,
\]

and, by Lemma 3, $(\partial/\partial u)^{k-\nu} \left[\phi(t) - \phi(u)\right]^{k-1}$ is a linear combination of expressions of the form

\[\left\{\phi(t) - \phi(u)\right\}^{k-1-\mu} \prod_{\nu=1}^{k-\nu} \phi^{(p)}(u)^{a_p},
\]
where \(0 \leq \mu \leq k - \nu - 1\) and
\[
\sum_{p=1}^{k-\nu} \alpha_p = \mu, \quad \sum_{p=1}^{k-\nu} p\alpha_p = k - \nu.
\]

Thus to prove (3.7) it is enough to show that
\[
\left(3.8\right) \quad \int_0^\omega \phi'(t) \left| \int_0^t A^{k-1}(u) \left\{ \phi(t) - \phi(u) \right\}^{k-1-\nu} \psi^{(\nu)}(u) \prod_{p=1}^{k-\nu} \left\{ \phi^{(p)}(u) \right\} \alpha_p du \right| dt
\]
\[
= o \left[ \left\{ \phi(\omega) \right\}^{(k-1)m+1} \right].
\]

Consider
\[
\left(3.9\right) \quad f(t) = \int_0^t A^{k-1}(u) \left\{ \phi(t) - \phi(u) \right\}^{k-1-\nu} \psi^{(\nu)}(u) \prod_{p=1}^{k-\nu} \left\{ \phi^{(p)}(u) \right\} \alpha_p du.
\]

Integrating (3.9) by parts and making use of the fact that summability \([R, \lambda, k, m]\) implies summability \((R, \lambda, k)\) we have
\[
f(t) = \left[ o\left(\psi^{(\nu)}(u) \right) \left\{ \phi(t) - \phi(u) \right\}^{k-1-\mu} \prod_{p=1}^{k-\nu} \left\{ \phi^{(p)}(u) \right\} \alpha_p \right]_0^t \]
\[
+ o \left( \int_0^t u^k \frac{d}{du} \left\{ \psi^{(\nu)}(u) \right\} \left\{ \phi(t) - \phi(u) \right\}^{k-1-\mu} \prod_{p=1}^{k-\nu} \left\{ \phi^{(p)}(u) \right\} \alpha_p du \right) \]
\[
= I_1 + I_2, \text{ say.}
\]

When \(0 \leq \mu < k - 1\), then \(I_1 = 0\); and when \(\mu = k - 1\),
\[
I_1 = \left[ o\left(\psi^{(\nu)}(t) \right) \prod_{p=1}^{k-\nu} \left\{ \phi^{(p)}(t) \right\} \alpha_p \right]
\]

Using hypotheses (ii) and (iii) and the relations
\[
\sum_{p=1}^{k-\nu} p\alpha_p = k - \nu, \quad \sum_{p=1}^{k-\nu} \alpha_p = \mu,
\]
we get
\[
I_1 = o \left( \prod_{p=1}^{k-\nu} \left\{ \gamma(t) \right\} p \left\{ \phi^{(p)}(t) \right\} \alpha_p \right)
\]
\[
= o \left( \left\{ \phi(t) \right\} \Sigma \alpha_p \right)
\]
\[
= o \left( \left\{ \phi(t) \right\}^\nu \right).
\]

Thus
\[(3.10) \quad I_1 = o\left(\left\{\phi(t)\right\}^{k-1}\right).\]

Now \(I_2 = I_3 + I_4 + I_5\), where

\[I_3 = o\left(\int_0^t u^k \{\phi(t) - \phi(u)\}^{k-2-\nu}\phi'(u)\psi^{(\nu)}(u) \prod_{p=1}^{k-\nu} \{\phi^{(p)}(u)\}^{\alpha_p} du\right),\]

\[I_4 = o\left(\int_0^t u^k \{\phi(t) - \phi(u)\}^{k-1-\nu}\psi^{(\nu+1)}(u) \prod_{p=1}^{k-\nu} \{\phi^{(p)}(u)\}^{\alpha_p} du\right),\]

\[I_5 = o\left(\int_0^t u^k \{\phi(t) - \phi(u)\}^{k-1-\nu}\psi^{(\nu)}(u) \sum_{p=1}^{k-\nu} \alpha_p \frac{\phi^{(p+1)}(u)}{\phi^{(p)}(u)} \prod_{r=1}^{k-\nu} \{\phi^{(r)}(u)\}^{\alpha_r} du\right).\]

By hypothesis (ii),

\[I_3 = o\left(\int_0^t \{\phi(t) - \phi(u)\}^{k-\nu}\psi^{(\nu)}(u) \prod_{p=1}^{k-\nu} \{\gamma(u)\}^{\nu_p} \phi^{(\nu_p)}(u) du\right)\]

\[= o\left(\{\phi(t)\}^\mu \int_0^t \{\phi(t) - \phi(u)\}^{k-\nu}\psi^{(\nu)}(u) du\right).\]

Thus

\[(3.11) \quad I_3 = o\left(\left\{\phi(t)\right\}^{k-1}\right).\]

Consider \(I_4\) (a) If all \(\alpha_p\)'s are zero then \(\mu = 0\) and \(k = \nu\), since \(\sum \alpha_p = \mu\) and \(\sum \nu p = k - \nu\). Thus

\[I_4 = o\left(\int_0^t t^k \{\phi(t) - \phi(u)\}^{k-1}\psi^{(k+1)}(u) du\right)\]

\[= o\left(t^k \{\phi(t)\}^{k-1-\nu} \int_0^t \psi^{(k+1)}(u) du\right)\]

\[= o\left(\{\phi(t)\}^{k-1}\right),\]

since \(t^k\psi^{(k)}(t) = O(1)\) by (ii).

(b) Suppose that not all \(\alpha_p\)'s are zero, so that \(\mu \geq 1\). If \(\alpha_q \geq 1\), we have, by (ii) and (iii)

\[I_4 = o\left(\{\phi(t)\}^{k-2} \int_0^t \{\phi(t) - \phi(u)\}^{k-2-\nu}\gamma(u)\phi^{(\nu)}(u) du\right)\]

\[= o\left(\{\phi(t)\}^{k-2}\int_0^t \gamma(t)\phi^{(\nu)}(u) du\right)\]

\[= o\left(\{\phi(t)\}^{k-2} \int_0^t \phi^{(\nu)}(u) du\right)\]

\[= o\left(\{\phi(t)\}^{k-2}\int_0^t \phi^{(\nu)}(u) du\right)\]

Thus
(3.12) \[ I_4 = o\left( \left\{ \phi(t) \right\}^{k-1} \right). \]

When \( k > 1 \),

\[
I_5 = o \left( \int_0^1 \left\{ \phi(t) - \phi(u) \right\}^{k-1-u} \sum_{p=1}^{k-p} \alpha_p \left\{ \gamma(u) \right\}^p | \phi^{(p+1)}(u) | \prod_{r=1}^{k-p} \left\{ \gamma(u) \right\}^r \phi^{(r)}(u) | \alpha_r du \right),
\]

where \( \prod^{(p)} \) means that in the \( p \)th factor the power is \( \alpha_p - 1 \) (not \( \alpha_p \)). Hence

\[
I_5 = o \left( \int_0^1 \left\{ \phi(t) - \phi(u) \right\}^{k-1-u} \sum_{p=1}^{k-p} \left\{ \gamma(u) \right\}^p \phi^{(p+1)}(u) \left\{ \phi(u) \right\}^{\alpha_p-1} du \right)
\]

\[
= o \left( \left\{ \phi(t) \right\}^{k-2} \sum_{p=1}^{k-p} \left\{ \gamma(t) \right\}^p \int_0^1 \phi^{(p+1)}(u) du \right)
\]

\[
= o \left( \left\{ \phi(t) \right\}^{k-2} \sum_{p=1}^{k-p} \left[ \left\{ \gamma(t) \right\}^p \phi^{(p)}(t) \right] \right)
\]

\[
= o \left( \left\{ \phi(t) \right\}^{k-1} \right).
\]

When \( k = 1 \), the proof is similar to that for \( I_4 \), case (a). Thus

(3.13) \[ I_5 = o\left( \left\{ \phi(t) \right\}^{k-1} \right). \]

From (3.10), (3.11), (3.12) and (3.13) it follows that the function \( f(t) \) of (3.9) is \( o\left( \left\{ \phi(t) \right\}^{k-1} \right) \). Thus

\[
\int_0^\omega \phi'(t) | f(t) | m dt = o \left( \int_0^\omega \phi(t) \left( (k-1)m\phi'(t) dt \right) \right)
\]

\[
= o \left( \left\{ \phi(t) \right\} \left( (k-1)m+1 \right) \right).
\]

This completes the proof of the theorem.

In the special case when \( \psi(t) = 1, \gamma(t) = t, m = 1 \), we have the "second theorem of consistency for strong summability" due to Srivastava [6].

When \( \phi(t) = e^t, \psi(t) = t^{-k}, \gamma(t) = 1, m = 1 \), we have the following strong Riesz summability analogue of a theorem due to Tatchell [7].

**Theorem.** If \( k \) is a positive integer and the series \( \sum a_n \) is summable \([R, \lambda, k]\), then \( \sum a_n \lambda^{-k} \) is summable \([R, e^t, k]\).

**References**


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