FIXED POINTS AND STABLE SUBGROUPS OF ALGEBRAIC GROUP AUTOMORPHISMS[1]

D. J. WINTER

1. **Introduction.** In this paper, we study the fixed point sets and stable subgroups of automorphisms of a connected algebraic linear group over an algebraically closed field of arbitrary characteristic. [7] contains a description of most of the results of this paper, and some material in [7] is essential for the present paper. Many of the results discussed in the present paper were proved at characteristic 0 by Borel-Mostow, Jacobson and Mostow (see [1], [5], [6]).

In §3 it is essentially shown that if σ_1, σ_2 are "semisimple" automorphisms of a connected linear algebraic group G which differ by an inner automorphism of G, and if H_i is a Cartan subgroup of the connected component of the subgroup of fixed points of σ_i $(i=1, 2)$, then there is an inner automorphism $\text{Int}\, g$ of G such that $H_1 \,\text{Int}\, g = H_2$ and $\text{Int}\, g^{-1} \circ \sigma_1 \circ \text{Int}\, H_1 \circ \text{Int}\, g = \sigma_2 \circ \text{Int}\, H_2$. This is used (see §4) to establish that a semisimple automorphism of a connected algebraic linear group G stabilizes some Borel subgroup of G and some Cartan subgroup of G; and that a semisimple automorphism of a connected semisimple algebraic group G keeps fixed some regular element of G.

2. **Preliminaries.** Throughout this paper, G is a connected linear algebraic group over an algebraically closed field of arbitrary characteristic 0 or p. We do not distinguish between G and birational isomorphic images of G. Some familiarity with the general structure theory of algebraic groups is assumed in the discussion (see [3]).

A (birational) automorphism σ of G is *algebraic* if G can be birationally identified with a closed normal subgroup of an algebraic linear group K over F which contains an element s such that $\sigma = \text{Int}_G\, s$ ($\text{Int}_G\, s$ denotes the automorphism of G sending an element g of G into $s^{-1}gs$). Such a σ is said to be *semisimple* (*unipotent*) if for some such K and s, s is semisimple (unipotent). (This terminology is equivalent to that of [7], as is noted in [7].)

An algebraic automorphism σ has a unique decomposition $\sigma = \sigma_s\sigma_u$ where σ_s, σ_u are commuting algebraic automorphisms which are

Received by the editors August 13, 1966.

[1] This paper was written while the author was supported by the Army Research Office, Durham. It is a reformulation of material contained in a dissertation directed by Professor George Seligman, presented for the degree of Doctor of Philosophy at Yale University, and written on a National Science Foundation Fellowship.

respectively semisimple and unipotent. Every (birational) automorphism of a connected semisimple algebraic linear group over F is algebraic. (The above facts are proved in [**7**].)

The Zariski-connected component of the identity of a subgroup S of G is denoted by S_0. The set of fixed points of an automorphism σ of G is denoted by $F_G(\sigma)$ (this is a slight departure from the terminology of [**7**]). If K is an algebraic linear group containing G as a closed normal subgroup and if g, h are elements of K, we use the following notation: $g^h = h^{-1}gh$; $C_G(h) = \{g \text{ in } G \mid gh = hg\}$; h_s, h_u denote respectively the semisimple and unipotent parts of h.

3. **Cartan subgroups of** $F(\sigma)_0$. In this section it is in effect shown that if σ is a semisimple automorphism of G, then properties of a Cartan subgroup of $F(\sigma)_0$ relative to the action of G on itself given by $g\colon x \to \sigma(g^{-1})xg$ are roughly the same as properties of a Cartan subgroup of G relative to the action of G on itself given by $g\colon x \to g^{-1}xg$. This is then used to establish the following theorem, which is needed for the material discussed in the next section:

THEOREM 1. *Let σ and τ be semisimple elements of a linear algebraic group K containing G as a closed normal subgroup. Suppose that $\sigma G = \tau G$. Let H be a Cartan subgroup of $C_G(\sigma)_0$, L a Cartan subgroup of $C_G(\tau)_0$. Then there exists g in G such that $H^g = L$ and $(\sigma H)^g = \tau L$.*

LEMMA 1. *Let K be an algebraic group. Let G, R be closed connected normal subgroups of K with $R \subseteq G$ and R solvable. Let σ be a semisimple element of K such that $g^\sigma R = gR$ for g in G. Then $G = C_G(\sigma)_0 R$.*

PROOF. Suppose first that R is a torus. Let B be a Borel subgroup of G. $B \supseteq R$ (since BR is a solvable connected subgroup of G) and B has the form $B = TU$ where T is a maximal torus of B and U is a closed normal unipotent subgroup of B (see [**3**, §6-06]). As a normal torus in B, R is central in B (see [**3**, §4-07]) and hence is contained in T. Thus $T^\sigma = T$. And $\sigma | U = id_U$ (since for u in U, $u^\sigma = us$ for some s in R; and $us = su$ since R is central in B, so that $s = 1$ since u^σ is semisimple). We claim that $T = C_T(\sigma)R$ (it then follows that $B \subseteq C_G(\sigma)R$). Suppose first that σ has finite order k. Let $\mathfrak{g} = \langle\sigma\rangle$ be the cyclic group generated by σ and let l be an integer with $(k, l) = 1$. For any torus S, we define $S_l = \{s \text{ in } S \mid s^l = 1\}$. Then T_l and R_l are $\mathfrak{g}$-stable. And for t in T_l, τ in $\mathfrak{g}$, there exists a unique s_τ in R_l such that $t^\tau = ts_\tau$. The mapping s_τ from $\mathfrak{g}$ into R_l is a 1-cocycle, as is easily checked. Since $(k, l) = 1$, $H^1(\mathfrak{g}, R_l) = \{0\}$ and there exists c in R_l such that $c(c^{-1})^\tau = s_\tau$ for τ in $\mathfrak{g}$. For such a c, $(tc)^\sigma = ts_\sigma c^\sigma = tc$. Thus

$T_l \subseteq C_G(\sigma)R$. Since $\bigcup_{(k,l)=1} T_l$ is dense in T (see [3, §6-13]), it follows that $T \subseteq C_G(\sigma)R$. Thus $B \subseteq C_G(\sigma)R$. Since every element of G is contained in some Borel subgroup of G, it follows that $G \subseteq C_G(\sigma)R$. Thus $G \subseteq C_G(\sigma)_0 R$.

If σ is not finite, let A be the closed subgroup of K generated by σ. Then $A = \bigcup_{i=1}^m A_0\sigma^i$ where $m = (A : A_0)$. Choose a_j in A_0 such that $\langle a_j\rangle \subseteq \langle a_{j+1}\rangle$ for all j, $\bigcup\langle a_j\rangle$ is dense in A_0, and a_j has finite order relatively prime to m for all j (see [3, §6-13]). Choose a in A_0 such that $a^m = \sigma^m$ (this is possible since A_0 is a torus). Let $\tau = \sigma a^{-1}$. Then τ has order m and the groups $\mathfrak{g}_j$ generated by τ and a_j are finite cyclic groups (since A is abelian and a_j has finite order relatively prime to m). Since $A = \bigcup_{i=1}^m A_0\tau^i$, $\bigcup \mathfrak{g}_j$ is dense in A. Choose j such that $C_G(\mathfrak{g}_j) = C_G(A) = C_G(\sigma)$. Since $\mathfrak{g}_j$ is generated by a single semisimple element of K of finite order, we have $G \subseteq C_G(\mathfrak{g}_j)_0 R = C_G(\sigma)_0 R$ by previous work.

We now consider the general case, R solvable. Let V be the (closed characteristic connected) subgroup of unipotent elements of R (see [3, §6-06]). Let $\bar{\sigma}$ be the automorphism of $\overline{G} = G/V$ induced by σ. Since $\overline{R} = R/V$ is a normal torus in $\overline{G}$, we have $\overline{G} = C_{\overline{G}}(\bar{\sigma})\overline{R}$. To show that $G = C_G(\sigma)R$, it therefore suffices to show that if g in G satisfies $g^\sigma V = gV$, then g is in $C_G(\sigma)V$. This is true since σ is semisimple (Corollary 1, §6-01 of [3]). Thus $G = C_G(\sigma)R$, whence $G = C_G(\sigma)_0 R$.

For the rest of §3, K is an algebraic linear group containing G as a closed normal subgroup; σ is a semisimple element of K; G acts on σG, an element of h of G sending σg into $(\sigma g)^h = h^{-1}\sigma g h$ for g in G; and if V is a subvariety of K, we let $N(V) = N_G(V) = \{g \text{ in } G \mid V^g = V\}$.

PROPOSITION 1. *Let H be a Cartan subgroup of $C_G(\sigma)_0$. Then $N(\sigma H)_0 = N(\sigma T)_0 = H$, where $T = H_s = \{h_s \mid h \in H\}$.*

PROOF. We recall that T is the unique maximal torus of H (see [3, §6-04]). Clearly $H \subseteq N(\sigma H)$. And $N(\sigma H) \subseteq N(\sigma T)$ because $(\sigma H)_s = \sigma H_s = \sigma T$ and the action of G on G stabilizes the subset of semisimple elements. It therefore suffices to show that $N(\sigma T)_0 \subseteq H$. Now $N(\sigma T) = \{g \in N(T) \mid g^\sigma T = gT\}$ (e.g. if $(\sigma T)^g = \sigma T$, then $\sigma^g T^g = \sigma T$ and consequently $\sigma^g T = \sigma T$ and $T^g = T$; the latter implies that T contains $(\sigma^{-1})^g\sigma = g^{-1}\sigma^{-1}g\sigma$, whence $gT = g^\sigma T$). Using this and Lemma 1, we have $N(\sigma T)_0 = C_{N(\sigma T)_0}(\sigma)T \subseteq C_{N(T)_0}(\sigma)$. But $N(T)_0 \subseteq C_G(T)$ [3, §6-04]. Thus $N(\sigma T)_0 \subseteq (C_G(T) \cap C_G(\sigma))_0 = H$ (see [3, §7-01]).

DEFINITION. An element x in σG is *V-isolated* (for $V \subseteq \sigma G$) if x is contained in only finitely many distinct sets of the form V^g (g in G).

PROPOSITION 2. *Let H be a Cartan subgroup of $C_G(\sigma)_0$ and let $T = H_s$. Then for some t in T, σt is σT-isolated and σH-isolated.*

PROOF.[2] (a) We first show that if t is an element of T of finite order m relatively prime to $(\langle\bar{\sigma}\rangle: (\langle\bar{\sigma}\rangle)_0)$, then $(\sigma t)^g \in \sigma H$ implies that $(\sigma\langle t\rangle)^g \subseteq \sigma T$. Thus suppose that t satisfies the above hypothesis. Then $((\sigma t)^g)^m = (\sigma^m t^m)^g = (\sigma^m)^g$ is contained in $\sigma^m H$. Thus if $\tau = \sigma^m$, it follows that $(\tau^i)^g \in \tau^i H$ for i an integer. Thus $\delta^{-1}\delta^g \in H$ for δ in $\langle\tau\rangle$ and therefore for δ in $\langle\bar{\tau}\rangle$ (since $\delta \to \delta^{-1}\delta^g$ is continuous on K and H is closed). Since $\langle\bar{\tau}\rangle \subseteq \langle\bar{\sigma}\rangle$ and $\sigma^m \in \langle\bar{\tau}\rangle$, $(\langle\bar{\sigma}\rangle: \langle\bar{\tau}\rangle)$ divides m and $\langle\bar{\tau}\rangle \supseteq \langle\bar{\sigma}\rangle_0$ (for $\bigcup_{i=1}^m \langle\bar{\tau}\rangle\sigma^i$ is a closed group containing σ and thus equals $\langle\bar{\sigma}\rangle$). Since m is relatively prime to $(\langle\bar{\sigma}\rangle: \langle\bar{\sigma}\rangle_0)$, it follows that $\langle\bar{\sigma}\rangle = \langle\bar{\tau}\rangle$ and $\sigma \in \langle\bar{\tau}\rangle$. Thus $\sigma^{-1}\sigma^g \in H$ and $\sigma^g \in \sigma H$.

Since $(\sigma t)^g = \sigma^g t^g \in \sigma H$ and $\sigma^g \in \sigma H$, we must have $t^g \in H$. Consequently we have $t^g \in T$. Thus $\langle t\rangle^g \subseteq T$. Thus $(\sigma\langle t\rangle)^g \subseteq \sigma T$, since $\sigma^g \in \sigma H$ implies $\sigma^g \in \sigma T$.

(b) Let $d = \dim T$. Choose a set π of d distinct primes not dividing $(\langle\bar{\sigma}\rangle: \langle\bar{\sigma}\rangle_0)$ or p. Then following [**3**, §6-13], we can find a finite cyclic subgroup $\langle t\rangle$ in T whose order is divisible by a prime q only if $q \in \pi$, and having the property that $\sigma^{-1}g^{-1}\sigma\langle t\rangle g \subseteq T$ if and only if $\sigma^{-1}g^{-1}\sigma Tg \subseteq T$. For such a t, $g^{-1}\sigma t g \in \sigma H$ implies that $g^{-1}\sigma\langle t\rangle g \subseteq \sigma T$ by (a), and $g^{-1}\sigma\langle t\rangle g \subseteq \sigma T$ implies that $(\sigma T)^g = \sigma T$ by the above choice of t. Thus $(\sigma t)^g \in \sigma H \Leftrightarrow (\sigma T)^g = \sigma T$.

We claim that for such a t, σt is σT-isolated and σH-isolated. σt is σT-isolated because $\sigma t \in (\sigma T)^g$ implies that $(\sigma t)^{g^{-1}} \in \sigma T \subseteq \sigma H$ which implies that $(\sigma T)^g = \sigma T$. We next show that σt is σH-isolated. Choose $S \subseteq G$ such that $\sigma t \in (\sigma H)^s$ for s in S; and such that if $\sigma t \in (\sigma H)^g$, then $(\sigma H)^g = (\sigma H)^s$ for a unique s in S. We must show that S is finite. If s is in S, then $(\sigma t)^{s^{-1}} \in \sigma H$, so that $(\sigma T)^{s^{-1}} = \sigma T$. Thus $S \subseteq N(\sigma T)$. For distinct r, s in S, we have $(\sigma H)^r \neq (\sigma H)^s$ or $r^{-1}s \notin N(\sigma H)$. Thus card $S \leqq (N(\sigma T): N(\sigma H)) \leqq (N(\sigma T): N(\sigma T)_0)$ since $N(\sigma H)_0 = N(\sigma T)_0$. Thus S is finite and σt is σH-isolated.

PROPOSITION 3. *Let O contain a dense open subset of H. Then $(\sigma O)^G$ contains a dense open subset of σG.*

PROOF. It is easily shown that $(\sigma O)^G$ contains an open dense subset of $(\sigma H)^G$. And $(\sigma H)^G$ contains an open dense subset of $[(\sigma H)^G]^-$. Since $N(\sigma H)_0 = H$ (Proposition 1) and since σH contains a σH-isolated point (Proposition 2), $\dim [(\sigma H)^G]^- = \dim \sigma H + \dim G - \dim H = \dim G$. (Here we use a very close analog of Lemma 5 of §6-11 in [**3**] whose validity is easily seen from the methods of [**3**, §6-12] and Theorem 2 on p. 106 of [**2**]; details are given in [**8**, pp.

[2] In this proof we use the misleading notation $\langle\bar{\sigma}\rangle$ for the closed subgroup generated by σ.

22–24].) It follows that $[(\sigma H)^G]^- = G$ and therefore that $(\sigma O)^G$ contains an open dense subset of G.

LEMMA 2. *Let $S = \{t \text{ in } T \mid \sigma t \text{ is } \sigma T\text{-isolated}\}$. Then S contains an open dense subset of T.*

PROOF. Let B denote the set of σT-isolated points of $[(\sigma T)^G]^-$. Then B contains an open dense subset of $[(\sigma T)^G]^-$. (This is also clear from Theorem 2, p. 106 of [2] and the methods of [3]—see corollary of §6-12; details are given in [8, pp. 22–24].) Since $(\sigma T)^G$ is épais, it follows that $B \cap (\sigma T)^G$ contains an open dense subset of U of $[(\sigma T)^G]^-$. Let x be an element of U. Then $x = (\sigma t)^g$ for some t in T, g in G. Now $\sigma t = x^{g^{-1}} \in U^{g^{-1}} \subseteq B$. Thus $\sigma T \cap U^{g^{-1}}$ is a nonempty open (hence dense) subset of σT which is contained in σS.

LEMMA 3. *Let σt in σT be σT-isolated. Then $C_G(\sigma t)_0 = H$.*

PROOF. Let g be an element of $C_G(\sigma t)$. Then $\sigma t = (\sigma t)^g \in (\sigma T)^g$. Thus the set $\mathcal{S} = \{(\sigma T)^g \mid g \in C_G(\sigma t)\}$ of conjugates of the variety σT is finite, σt being σT-isolated. Therefore the isotropy group $C_G(\sigma t) \cap N(\sigma T)$ in $C_G(\sigma t)$ of the element σT of $\mathcal{S}$ is a closed subgroup of $C_G(\sigma t)$ of finite index. Thus $C_G(\sigma t)_0 \subseteq C_G(\sigma t) \cap N(\sigma T)$. Thus $C_G(\sigma t)_0 \subseteq N(\sigma T)_0 = H$. On the other hand, it is obvious that $H \subseteq C_G(\sigma t)_0$.

PROOF OF THEOREM 1. Regard G as a transformation group on σG. Let T be the set of semisimple elements of H and let $S = \{t \in T \mid \sigma t \text{ is } \sigma T\text{-isolated}\}$. Let $O_H = SU$ where U is the (closed normal connected) subgroup of unipotent elements of H and recall that $su = us$ for s in S, u in U [3, §6-06, §4-07]. Then O_H contains an open dense subset of H by Lemma 2; and $H = C_G(\sigma h_s)_0$ for h in O_H by Lemma 3. There is a similar O_L for (τ, L). By Proposition 3, $(\sigma O_H)^G$ and $(\tau O_L)^G$ contain open dense subsets of σG. Thus $(\sigma O_H)^G \cap (\tau O_L)^G$ is nonempty and there exist g_1, g_2 in G, h in O_H, l in O_L, such that $g_1^{-1}\sigma h g_1 = g_2^{-1}\tau l g_2$. Then $g_1^{-1}\sigma h_s g_1 = g_2^{-1}\tau l_s g_2$. Taking connected centralizers in G, we have $g_1^{-1}Hg_1 = g_2^{-1}Lg_2$. The assertions of the theorem follow immediately.

4. Fixed points and stable subgroups of σ.

The following theorem is shown in [7] to be a consequence of the preceding theorem:

THEOREM 2. *Let G be a connected semisimple algebraic group. Let σ be a semisimple (algebraic) automorphism of G. Then σ keeps stable a Borel subgroup B of G and a maximal torus T of G contained in B. For any such B and T, $F_T(\sigma)_0$ is a Cartan subgroup of $F_G(\sigma)_0$. $F_G(\sigma)_0$ contains a regular element of G.*

THEOREM 3.[3] *Let σ be a semisimple algebraic automorphism of a connected algebraic group G. Then σ keeps stable a Borel subgroup of G. Moreover the centralizer in G of a maximal torus in $F_G(\sigma)_0$ is solvable.*

PROOF. Let R be the radical of G. Then σ induces an automorphism $\bar{\sigma}$ on $\overline{G}=G/R$. $\bar{\sigma}$ is clearly a semisimple algebraic automorphism of $\overline{G}$. By Theorem 2, $\bar{\sigma}$ stabilizes a Borel subgroup of $\overline{G}$. The latter must have the form $\overline{B}=B/R$ where $B\supseteq R$ and B is a σ-stable subgroup of G.

A maximal torus of $F_{\overline{G}}(\bar{\sigma})_0$ has the form $\overline{C}=C/R$ where $C\geqq R$. By Lemma 1, $C=F_C(\sigma)_0R$. Since $F_C(\sigma)_0$ is solvable, $F_C(\sigma)_0=SU$ where S is a torus and U is a closed connected unipotent subgroup. Since C/R is a torus in $\overline{G}$, $U\subseteq R$. Thus $C=SR$. By Theorem 8, $F_{\overline{G}}(\bar{\sigma})_0$ contains a regular element of $\overline{G}$. Thus the same is true of $\overline{C}$ and $C_{\overline{G}}(\overline{C})$ is abelian. (The centralizer of a torus in a connected algebraic group is connected [3, §6–14].) Thus $C_G(S)$ is solvable (since $C_G(S)R/R\subseteq C_{\overline{G}}(\overline{C})$). This establishes that the centralizer of a maximal torus in $F_G(\sigma)_0$ is solvable, since any such torus contains a conjugate of S.

LEMMA 4. *Let G be a connected solvable algebraic group. Let $\mathfrak{g}$ be a finite group of birational automorphisms of G. Then if $\mathfrak{g}$ has order relatively prime to p, $\mathfrak{g}$ keeps stable a Cartan subgroup of G.*

PROOF. Let $G=TU$ be the usual decomposition of G as product of a torus T and a closed connected characteristic unipotent subgroup U. U is $\mathfrak{g}$-stable. An easy induction argument shows that we can assume without any loss of generality that U is abelian and that $C_U(G)_0=\{1\}$. Since G is solvable, $N_U(T)=C_U(T)$ and both are connected [3, §6-04]. Since U is abelian, $C_u(T)=C_U(G)$. Thus $N_U(T)=N_U(T)_0=C_U(G)_0=\{1\}$ and $N_U(T)=\{1\}$. Thus for σ in $\mathfrak{g}$, there is a *unique* u_σ in U such that $T^\sigma=T^{u_\sigma}$. The mapping u_σ from $\mathfrak{g}$ into U is clearly a 1-cocycle. Since U is uniquely m-divisible where m is the order of $\mathfrak{g}$, $H^1(\mathfrak{g}, U)=\{0\}$ and there exists v in U such that $v(v^{-1})^\sigma=u_\sigma$ for σ in $\mathfrak{g}$. It is seen by a simple computation that T^v is $\mathfrak{g}$-stable. Thus so is the Cartan subgroup $C(T)^v$.

THEOREM 4. *Let σ be a semisimple algebraic automorphism of a connected algebraic group G. Then σ keeps stable a Cartan subgroup of G.*

PROOF. By Theorem 3, σ keeps stable a Borel subgroup of G. Thus with no loss of generality we may assume that G is solvable. We may assume that G is a closed connected normal subgroup of K and that s is a semisimple element of K such that $\sigma=\mathrm{Int}_G\, s$. Let S be the closed

[3] R. Steinberg has proved independently, and under weaker conditions on σ, that σ stabilizes a Borel subgroup of G.

subgroup generated by s. Then S is diagonalizable and $S=\mathfrak{g}S_0$ for some finite subgroup $\mathfrak{g}$ of S (e.g. see part of the proof of Lemma 1). Now the connected centralizer of S_0 in $\mathfrak{g}$ has a $\mathfrak{g}$-stable maximal torus T by Lemma 4. Clearly $S_0 \subseteq T$ and therefore T is a maximal torus of G. T is clearly S-stable. Thus T is σ-stable. Thus σ stabilizes the Cartan subgroup $C(T)$.

Bibliography

1. A. Borel and G. D. Mostow, *On semi-simple automorphisms of Lie algebras*, Ann. of Math. **61** (1955), 389–504.

2. C. Chevalley, *Fondements de la géométrie algébrique*, Faculté des Sciences de Paris, 1958.

3. C. Chevalley (Séminaire C. Chevalley), *Classification des groupes de Lie algébrique*, Vols. I, II, École Norm. Sup., Paris, 1958.

4. G. Hochschild and G. D. Mostow, *Representations and representative functions of Lie groups*, Ann. of Math. **66** (1957), 495–542.

5. N. Jacobson, *A note on automorphisms of Lie algebras*, Pacific J. Math. **12** (1962), 303–315.

6. G. D. Mostow, *Fully reducible subgroups of algebraic groups*. Amer. J. Math. **78** (1956), 200–221.

7. D. J. Winter, *On automorphism of algebraic groups*, Bull. Amer. Math. Soc. **72** (1966), 706–708.

8. ———, *On automorphisms of Lie algebras and algebraic groups*, Doctoral Dissertation, Yale Univ., New Haven, Conn., 1966.

Yale University