A VERSION OF THE LÉVY-BAXTER THEOREM FOR
THE INCREMENTS OF BROWNIAN MOTION
OF SEVERAL PARAMETERS

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1. Introduction. Let \( X(t_1, \ldots, t_k), -\infty < t_1, \ldots, t_k < \infty, \) be Lévy's Brownian motion process of \( k \) parameters: it is a Gaussian process with mean 0 and covariance function

\[
(1.1) r(s, t) = \text{EX}(s_1, \ldots, s_k)X(t_1, \ldots, t_k) = \frac{1}{2} \left( \| s \| + \| t \| - \| t - s \| \right)
\]

where \( t = (t_1, \ldots, t_k) \) and \( \| t \| = (t_1^2 + \cdots + t_k^2)^{1/2} \). For each integer \( n \geq 1 \), the unit cube \( \{ t: 0 \leq t_1 \leq 1, \ldots, 0 \leq t_k \leq 1 \} \) can be broken up into \( 2^{nk} \) cubes whose edges have the common length \( 2^{-n} \). Such cubes have corner-points of the form \( (i_12^{-n}, \ldots, i_k2^{-n}) \), where the \( i \)'s are integers between 0 and \( 2^n \). Put \( i = (i_1, \ldots, i_k) \). Let us denote by \( Y_{i,n} \) the \( k \)th-order difference of the sample function \( X \) over the cube \( \{ t: (i_1-1)2^{-n} \leq t_1 \leq i_22^{-n}, \ldots, (i_k-1)2^{-n} \leq t_k \leq i_k2^{-n} \} = C(i, n) \):

\[
Y_{i,n} = \Delta_1 \cdots \Delta_k X = X(i_22^{-n}, \ldots, i_k2^{-n}) - \sum_{r=1}^{k} p_r + \sum_{r<s} p_{rs} \nonumber
\]

\[
- \cdots + (-1)^k X((i_1-1)2^{-n}, \ldots, (i_k-1)2^{-n}) \nonumber
\]

where \( p_{rs} \ldots t \) denotes \( X(c_1, \ldots, c_k) \) for \( c_r = (i_r-1)2^{-n}, \ldots, c_t = (i_t-1)2^{-n} \)

and the remaining \( c_j \) equal \( i_j2^{-n} \).

The result we shall prove is

**Theorem.** For \( n \geq 1 \), let \( \sum | Y_{i,n} |^{2k} \) be the sum of the \( 2k \)th powers of the \( Y_{i,n} \) over all cubes \( C(i, n) \). Its limit, for \( n \to \infty \), exists with probability 1 and is equal to a numerical constant \( B_k \).

This represents a generalization of a classical theorem on the increments of the Brownian motion process of a one-dimensional time parameter, due to Lévy [2]. One can also extend this to a generalization of Baxter's theorem [1] to more general Gaussian processes of several parameters: indeed the proof of our theorem depends on the explicit form of the covariance only through the estimate in Lemma

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Received by the editors October 4, 1966.

1 This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the National Science Foundation, Grant NSF-GP-6237.

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2.2 and the fact that the variance of \( Y_{i,n} \) is bounded by a constant multiple of \( 2^{-n} \).

2. Preliminary lemmas.

**Lemma 2.1.** Let \( X \) and \( Y \) be random variables with a joint Gaussian distribution having means 0, common variance \( \sigma^2 \), and correlation coefficient \( \rho \); then

\[
E\{X^{2k}Y^{2k}\} - E\{X^{2k}\}E\{Y^{2k}\} \leq K\rho^2\sigma^{4k}
\]

where \( K \) is some numerical constant depending only on \( k \).

**Proof.** It is sufficient to prove the inequality for the special case \( \sigma = 1 \), as the general case follows from it. Suppose that a random variable \( U \) has a Gaussian distribution with mean \( m \) and variance \( s^2 \). An elementary computation shows that

\[
E[U^{2k}] = \sum_{j=0}^{k} c_j m^{2j}s^{2(k-j)}
\]

for some constants \( c_0, c_1, \cdots, c_k \) with \( c_0 = (2k)!/k!2^k \). The conditional distribution of \( X \) given \( Y \) is that of \( U \) for \( m = \rho Y, s^2 = 1 - \rho^2 \); therefore,

\[
E[X^{2k} \mid Y] = \sum_{j=0}^{k} c_j \rho^{2j} Y^{2j}(1 - \rho^2)^{2(k-j)}.
\]

Multiply each side of this equation by \( Y^{2k} \) and take expectations:

\[
E[X^{2k}Y^{2k}] = \sum_{j=0}^{k} c_j \rho^{2j} (1 - \rho^2)^{2(k-j)} \frac{[2(k+j)]!}{(k+j)!2^{k+j}}
\]

\[
= \left\{ \frac{(2k)!}{k!2^k} \right\}^2 \left\{ 1 - 2kp^2 + P(\rho) \right\}
\]

where \( P(x) \) is a polynomial whose terms of degrees 0 and 1 have coefficients 0. The assertion of the lemma now follows from the inequality \( |\rho|^k \leq \rho^2 \) for \( |\rho| \leq 1 \) and \( k \geq 2 \).

**Lemma 2.2.** Put \( r = (x_1^2 + \cdots + x_k^2)^{1/2} \); then

\[
\left| \frac{\partial^{2k} r}{\partial x_1^2 \cdots \partial x_k^2} \right| \leq \text{constant} \frac{1}{r^{2k-1}}
\]

for all values of \( x_1, \cdots, x_k \).

**Proof.** There are constants \( c_{0k}, \cdots, c_{kk} \) such that

\[
\frac{\partial^{2k} r}{\partial x_1^2 \cdots \partial x_k^2} = \sum_{j=0}^{k} c_{jk} r^{-(2(k+j)-1)} \sum_{1 \leq i_1 < \cdots < i_j \leq k} x_{i_1}^2 \cdots x_{i_j}^2.
\]
where the summation over \( i_1, \cdots, i_j \) is understood to be 1 for \( j = 0 \). This expression can be verified by induction on \( k \); for this purpose, we note that derivatives of \( r \) with respect to \( x_1, \cdots, x_j \) depend on \( x_{j+1}, \cdots, x_k \) only through \( r \). It follows that the derivative on the left-hand side of the above equation is dominated by a constant multiple of

\[
- (2^{k-1}) \sum_{j=0}^{k} \sum_{1 \leq i_1 < \cdots < i_j \leq k} r^{-2j} x_{i_1}^2 \cdots x_{i_j}^2.
\]

The sum in the latter expression is bounded: for, on one hand, we have

\[
- 2j \frac{2^j}{r} x_{i_1}^2 \cdots x_{i_j}^2 \leq \frac{(x_{i_1}^2 \cdots x_{i_j}^2)^j}{(x_{i_1}^2 + \cdots + x_{i_j}^2)^j};
\]

and, on the other hand,

\[
\frac{2^j}{r} x_{i_1}^2 \cdots x_{i_j}^2 \leq j^{-j} (x_{i_1}^2 + \cdots + x_{i_j}^2)^j
\]

because the geometric mean never exceeds the arithmetic mean.

3. **Proof of the Theorem.** For \( n \geq 1 \), the random variables \( \{ Y_{i,n} \} \) have a joint Gaussian distribution. The means are all 0. Let us denote by \( D_k \) the variance of the \( k \)-th order difference of \( X(\cdot) \) over the corner-points of the unit cube, i.e., the variance of

\[
X(1, \cdots, 1) - X(0, 1, \cdots, 1) - \cdots - X(1, \cdots, 1, 0) + X(0, 0; 1, \cdots, 1) + \cdots \pm X(0, 0, \cdots, 0).
\]

The \( Y_{i,n} \) have a common variance equal to \( 2^{-n} D_k \): for, on one hand, the joint distribution of any finite collection of differences of the process \( X \) is invariant under translations of the parameter set, by virtue of the form (1.1) of the covariance function, and so \( Y_{i,n} \) has the same distribution as \( Y_{0,n} \); and, on the other hand, \( Y_{0,n} \) has the variance \( 2^{-n} D_k \) because the process \( X(ct_1, \cdots, ct_k) \) is stochastically equivalent to the process \( c^{1/2} X(t_1, \cdots, t_k) \), for any constant \( c > 0 \). The covariance of \( Y_{i,n} \) and \( Y_{j,n} \) is equal to the \( 2k \)-th order difference of the function \( -\frac{1}{2} || s - \ell || \) over the product of the cubes \( C(i, n) \) and \( C(j, n) \). If the latter cubes are disjoint, then the difference is representable as the integral

\[
- \frac{1}{2} \int_{C(i,n)} \int_{C(j,n)} \frac{\partial^{2k} || \ell ||}{\partial s_1 \cdots \partial s_k \partial t_1 \cdots \partial t_k} ds_1 \cdots ds_k dt_1 \cdots dt_k.
\]

For proving the theorem it suffices to show that

\[
E \left\{ \sum_i Y_{i,n}^{2k} \right\} = \frac{2k!}{k!} \frac{2^{k}}{12^k} D_k = B_k;
\]
\[ \sum_n \text{Variance}\left( \sum_i |Y_{i,n}|^{2k} \right) < \infty. \]

The first relation is directly deducible from the distribution of \( Y_{i,n} \); we shall shortly verify the second relation by showing that the variance of \( \sum_i |Y_{i,n}|^{2k} \) is dominated by a constant multiple of \( 2^{-n} \), \( n = 1, 2, \cdots \).

In the rest of the paper we omit the subscript \( n \) from \( Y_{i,n} \), writing it as \( Y_i \). The variance of \( \sum_i |Y_i|^{2k} \) is

\[
(3.2) \quad \sum_{ij} \{ E[Y_i^{2k} Y_j^{2k}] - E[Y_i^{2k}] E[Y_j^{2k}] \}.
\]

There are \( 2^{2nk} \) terms in this sum. Each term is dominated by a constant multiple of \( 2^{-2nk} \); in fact,

\[
E[Y_i^{2k} Y_j^{2k}] \leq E[Y_i^{4k}] = 2^{-2nk} D_k^{2k} \left( \frac{(4k)!}{(2k)!} \right)^{2k}.
\]

There are at most \( 2^{2n(k-1)} \) terms in the sum for which the indices are restricted by an equation of the form \( i_1 - j_1 = \alpha_1 \) for some integer \( \alpha_1 \). The total contribution of such terms to the sum cannot exceed a constant multiple of \( 2^{-n} \). The same is true of the total contribution of all terms whose indices satisfy at least one of the inequalities \( |i_1 - j_1| \leq 1, \cdots, |i_k - j_k| \leq 1 \). We complete the proof by showing that the contribution of all terms whose indices satisfy all of the inequalities

\[
(3.3) \quad |i_1 - j_1| > 1, \cdots, |i_k - j_k| > 1
\]

does not exceed a constant multiple of \( 2^{-nk} \).

Lemma 2.1 implies that the general term of the sum (3.2) is bounded by a constant multiple of

\[
2^{-2nk} (\text{correlation}(Y_i Y_j))^2 = 2^{-2n(k-1)} (\text{covariance}(Y_i Y_j))^2 \cdot \text{constant}.
\]

Lemma 2.2 and equation (3.1) imply that the covariance of \( (Y_i Y_j) \) is dominated by a constant multiple of

\[
2^{-2nk} \max \{ ||s - t||^{-(2k-1)} : s \in C(i, n), t \in C(j, n) \} = 2^{-n} \left( (|i_1 - j_1| - 1)^2 + \cdots + (|i_k - j_k| - 1)^2 \right)^{-(2k-1)/2};
\]

thus, the general term of (3.2) whose indices satisfy (3.3) is dominated by a constant multiple of

\[
2^{-2nk} \left( (|i_1 - j_1| - 1)^2 + \cdots + (|i_k - j_k| - 1)^2 \right)^{-(2k-1)}.
\]

The sum of all such terms is not greater than
This is dominated by
\[ 2^{-nk} \sum_{i_1, \ldots, i_k=1}^{2^n} \left( i_1^2 + \cdots + i_k^2 \right)^{-\alpha (2k-1)} \]
because the geometric mean of \( i_1^2, \ldots, i_k^2 \) does not exceed the arithmetic mean. The coefficient of \( 2^{-nk} \) in the above expression is bounded. The proof is complete.

I thank the referee for suggesting a few corrections to the first draft of this paper.

References


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