POINTS OF MINIMUM NORM ON SMOOTH 
SURFACES IN BANACH SPACES

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Theorem. Suppose E is a real Banach space, and \( \phi \) is a continuously 
Fréchet differentiable real valued function defined on E. Assume that 
for some c in \( \mathbb{R} \) there is a \( u_0 \) in \( \phi^{-1}(c) \) such that \( \| u_0 \| \leq \| u \| \) for all \( u \) in 
\( \phi^{-1}(c) \). Then \( \| \phi'(u_0) \cdot u_0 \| = \| \phi'(u_0) \| \| u_0 \| \).

Proof. We may assume that \( u_0 \) and \( \phi'(u_0) \) are not zero. Let 
\( K = \ker \phi'(u_0) \). It will first be shown that \( k \in K \) implies \( \| u_0 + k \| \geq \| u_0 \| \).

Choose \( u_1 \in E \) with \( \phi'(u_0) \cdot u_1 = 1 \) and let \( E_2 = \text{span}\{ u_1 \} \). Then 
\( E = K \times E_2 \) and any \( u \in E \) can be written \( u = (y, \alpha u_1) \) where \( y \in K \) and 
\( \alpha = \phi'(u_0) \cdot u \). Let \( u_0 = (y_0, \alpha_0 u_1) \). Since \( \phi'(u_0) \neq 0 \) we may apply the 
implicit function theorem (see [2]) to obtain a \( C^1 \)-function \( g : U_1 \rightarrow \mathbb{R} \) 
where \( U_1 \) is a convex open neighborhood of zero in \( K \) and \( g \) satisfies 
\( g(0) = 0, g'(0) = 0 \) and

\[ \phi(y_0 + h, \alpha_0 u_1 + g(h) u_1) = c \text{ for all } h \in U_1. \]

Let \( B = \{ u \in E : \| u \| < \| u_0 \| \} \). By assumption \( \phi^{-1}(c) \cap B = \emptyset \). Assuming 
\( (u_0 + K) \cap B \neq \emptyset \) we will obtain a contradiction. Suppose there 
is \( k \in K \) with \( \| u_0 + k \| < \| u_0 \| \). We may assume that \( k \in U_1 \) and 
\( g(tk) > 0 \) for \( 0 < t \leq 1 \). Then for some \( s \) with \( 0 < s < 1 \) we have that 
\( (y_0 + k, \alpha_0 u_1 + sg(k)) \in B \). Since \( B \) is convex \((y_0 + \sigma k, \alpha_0 u_1 + \sigma sg(k)) \in B \) 
for \( 0 < \sigma \leq 1 \) so \( g(\sigma k) \geq \sigma sg(k) \) for \( 0 < \sigma \leq 1 \). Therefore \( (g(\sigma k) - g(0))/\sigma \geq \sigma sg(h)/\sigma = sg(k) \neq 0 \). In other words \( g'(0) \cdot k \neq 0 \) and this is a contra-
diction.

Therefore \( \| u_0 + k \| \geq \| u_0 \| \) for all \( k \in K \), and it follows that \( u_0 \in K \) 
and \( \| \alpha u_0 + k \| \geq \| \alpha u_0 \| \) for all \( \alpha \in \mathbb{R} \) and \( k \in K \). Let \( \epsilon > 0 \) and choose 
v \in E with \( \| v \| = 1 \) and \( \phi'(u_0) \cdot v \geq \| \phi'(u_0) \| - \epsilon \). Then \( v = \alpha u_0 + k \), 
k \in K and we have that \( 1 = \| \alpha u_0 + k \| \geq \| \alpha \| \| u_0 \| \). Hence \( \| \phi'(u_0) \| - \epsilon \leq \| \phi'(u_0) \cdot v \| = \| \phi'(u_0) \cdot \alpha u_0 \| \leq \| \phi'(u_0) \cdot u_0 \| \| u_0 \| \), so 
\( \| \phi'(u_0) \| \| u_0 \| \leq \| \phi'(u_0) \cdot u_0 \| \). The reverse inequality is trivial, so the proof is complete.

Corollary 1. Let \( \phi, E, \) and \( u_0 \) be as in the theorem. Suppose \( E = F^* \) 
where \( F \) is separable and \( \phi(u_n) \rightarrow \phi(u_0) \) whenever \( u_n \cdot x \rightarrow u_0 \cdot x \) for all 
\( x \in E \). Then in addition to the above conclusion it follows that \( \phi'(u_0) \in F^* \).

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Proof. \( \phi'(u_0) \in F^* \) and to show that \( \phi'(u_0) \in F \) it is sufficient to show that \( \phi'(u_0) \cdot u_n \to 0 \) whenever \( u_n \cdot x \to 0 \) for all \( x \in F \). Assuming this is false we can find a sequence \( \{v_n\} \subset F^* \) with \( \|v_n\| = 1 \) and \( |\phi'(u_0) \cdot v_n| > \gamma > 0 \) for all \( n \) and some \( \gamma > 0 \). Let

\[
\alpha_n = \max \{|\phi(u_0 + tv_n) - \phi(u_0)| : 0 \leq t \leq 1\}.
\]

Then \( \alpha_n \to 0 \) and we have that \( |(\phi(u_0 + \beta_n v_n) - \phi(u_0))/\beta_n| \leq \alpha_n/\beta_n \to 0 \) where \( \beta_n = \alpha_n^{1/2} \). However

\[
|\phi(u_0 + v) - \phi(u_0) - \phi'(u_0) \cdot v|/\|v\| \to 0 \quad \text{as } \|v\| \to 0
\]

so that \( |((\phi(u_0 + \beta_n v_n) - \phi(u_0))/\beta_n) - (\phi'(u_0) \cdot v_n)| \to 0 \) as \( n \to \infty \) giving that \( |\phi'(u_0) \cdot v_n| \to 0 \) which is a contradiction.

Corollary 2. Assume the hypothesis of the above corollary and suppose that \( F = L^1[0,1] \) so \( F^* = L^\infty[0,1] \). Then \( u_0 \in L^\infty, \phi'(u_0) \in L^1 \) and we have that \( u_0(t) = \pm \|u_0\| \sgn \phi'(u_0)(t) \) almost everywhere where \( \phi'(u_0)(t) \neq 0 \). In particular, if \( \phi'(u_0)(t) \neq 0 \) almost everywhere, \( u_0 \) is a "bang-bang" type solution which often occurs in control theory.

Remarks. (a) The conclusion above may be phrased in another way. Namely, \( \phi'(u_0)/\|\phi'(u_0)\| \) is a support functional to the unit sphere in \( E \) at the point \( \pm u_0/\|u_0\| \). If the norm \( N(u) = \|u\| \) is differentiable (except at zero) then support functionals are unique and it follows that \( N'(u_0) = \pm \phi'(u_0) \). In this case the above theorem reduces to the Lagrange method of multiplier result.

(b) Corollary 2 is proved in [3, pp. 302–311], for a special class of constraints \( \phi \). The result there suggested the above theorem. In [3] the argument is basically the following: First extend \( \phi \) to \( L^p[0,1] \), \( 1 < p < \infty \). Then, since the norm in \( L^p \) is differentiable, a Lagrange multiplier argument applies to give a solution \( u_p \). \( u_0 \) is obtained by letting \( p \to \infty \).

The norm in \( L^\infty[0,1] \) is nowhere differentiable, and in fact cannot be approximated by a differentiable function [1]. Therefore in using a Lagrange multiplier argument in [3], the indirect approach via \( L^p \) was essential.

Bibliography


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